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ELEMENTARY MECHANICS

FOR

UPPER SCHOOL CLASSES IN HIGH SCHOOLS

BY

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TORONTO
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PREFACE.

This work aims to cover the course in Elementary Mechanics prescribed for the Upper School.

An effort has been made to combine an experimental with a mathematical treatment of the subject.

The apparatus illustrated in Figures 12, 22, 32, 34, 39, 50 and 65 has been selected from an excellent set of mechanical apparatus designed by Professor W. J. Loudon, of the Department of Physics, Toronto University, and Professor C. H. C. Wright, of the School of Practical Science, Toronto.

F. W MERCHANT.

LONDON, *July 12th, 1906.*

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CONTENTS.

ELEMENTARY MECHANICS

CHAPTER I.

VELOCITY.

1. Position.

Position is **relative,** not **absolute.** The position of a point P is determined when its **distance** and its **direction** from some other point, taken as a point of reference, are known.

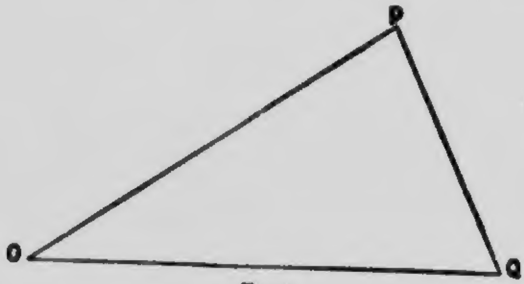

FIG. 1.

The line OP (Fig. 1) represents by its length and its direction the position of the point P with reference to the point O. Similarly, QP represents the position of the point P with reference to the point Q. That is, P has one position with reference to O, and another with reference to Q. In the same way, Q has one position, represented by the line OQ, with reference to O, and another, represented by the line PQ, with reference to P.

2. Motion.

A point is said to be in motion when its position is being changed continuously.

Motion, like position, is **relative.** A point is moving relatively to any point of reference when its position with respect to that point is changing continuously.

If the position of a point P with respect to a point of reference remains unchanged for a given time, P is said to be at rest with respect to this point of reference during that time; but, during the same time, the point F may be in motion with respect to another point of reference. A seat in a railway carriage in motion is at rest with respect to another seat in the same carriage, but in motion with respect to any object which has not the same motion as the carriage.

Two points are at rest relatively to each other when their motions are identical.

3. Displacement.

The change in the position of a point for any specified interval of time is called the **displacement** of the point for that interval.

Fig. 2.

If the point P is in motion relatively to the point A in the line AD (Fig. 2), and if at any instant it is at B, and at a subsequent instant at C, BC is the displacement of the point for the interval of time between these instants.

A displacement is determined when its magnitude and its direction are known.

The magnitude of the displacement is measured by the number of units of length contained in the line joining the two positions of the point.

If the point P has successive displacements BC, CD (Fig. 2) in the same direction, the total displacement equals

$$BC + CD,$$

the sum of these displacements; but if the point has displacements BC, CD, DE, EF (Fig. 2), some in one direction and some in the opposite direction, and the displacements in opposite directions are given opposite signs, the total displacement in the positive direction equals

$$BC + CD - DE - EF,$$

the algebraic sum of the displacements.

Velocity.

The amount of time required for a given displacement of a point depends upon the rapidity of the movement of the point. The time-rate at which the displacement takes place is, therefore, an important quantity.

The time-rate of motion of a body, without reference to its direction, is called **speed**. Thus, we speak of the speed of a horse or of a railway train without reference to direction of motion.

The time-rate of motion along a definite line, whose direction is given, is called velocity. In other words,

the term velocity includes the idea of rate of motion and that of its direction.

5. Uniform Velocity.

A point is said to be moving with a **uniform velocity** when it has equal displacements in equal intervals of time, however short these intervals may be.

6. Measure of Velocity.

A velocity, like a displacement, is determined when its magnitude and its direction are known. The magnitude of a velocity is measured by the number of times it contains some definite velocity, assumed as a unit. **The unit velocity is the velocity of a point which, moving uniformly, has a unit displacement in a unit of time.** The unit is a derived unit, because it is determined by the unit of length and the unit of time. For example, when the unit of length is the centimetre, and the unit of time the second, the unit of velocity is the centimetre per second.

With this system of units the measure of the velocity of a point moving uniformly is 1, when it has a displacement of 1 cm. in 1 second; 2, when it has a displacement of 2 cm. in 1 second; 3, when it has a displacement of 3 cm. in 1 second, and so on. If its displacement is 3 cm. in 2 seconds, in 1 second it is $\frac{3}{2}$ cm., and the measure of the velocity of the point is $\frac{3}{2}$.

In general terms, the measure of the velocity of a point moving uniformly, equals the measure of its displacement in a unit of time.

Thus, if s is the number of units of length in the displacement, and t the number of units of time in the

interval, the measure of the displacement in a unit of time is $\frac{s}{t}$. Then, if v is the measure of the velocity,

$$v = \frac{s}{t}$$

or

$$\mathbf{s} = \mathbf{t}\,\mathbf{v}.$$

That is, **the measure of the velocity** of a point moving uniformly during a given interval of time, **is obtained by dividing the measure of the displacement during that interval by the measure of the interval,**

or

the measure of the displacement equals the measure of the velocity multiplied by the measure of the interval.

7. Representation of Velocity.

The motion of a point at any instant is completely determined where its direction and rate are known. A velocity may, therefore, be completely represented by a straight line, because the direction can be represented by the direction of the line, and the magnitude by the length of the line, a certain unit length being taken to represent a unit velocity. For example, if one centimetre in length is taken to represent a velocity of one centimetre per second, the line A B (Fig. 3) which is 4 cm. in

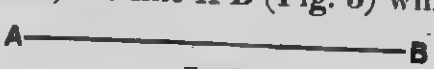

FIG. 3.

length, may be taken to represent a velocity of 4 cm. per second in a horizontal direction.

EXERCISE I.

1. Express in miles per hour a velocity of (1) 40 feet per second, (2) 100 yards per minute.

2. A point moves at the rate of 50 miles in $1\frac{1}{2}$ hours. What is its velocity in feet per second?

3. Find the ratio of velocities of (1) 60 miles per hour and 44 feet per second, (2) 5 miles per 6 minutes and 10 feet per $\frac{1}{4}$ second.

4. One body moves over 30 yards in 7 minutes, and another over 12 feet in 5 seconds. If their velocities are uniform, compare them.

5. How many times does the velocity 176 yards per hour contain the velocity 2 miles per minute?

6. A velocity of 20 miles per hour is v times a velocity of 30 feet per second. What is v?

7. How many times does a velocity of 120 metres per minute contain a velocity of 20 cm. per second?

8. What is the measure of a velocity of 400 cm. per second when the unit velocity is (1) a centimetre per second, (2) a centimetre per 2 seconds, (3) a centimetre per $\frac{1}{2}$ second, (4) 2 centimetres per second, (5) $\frac{1}{2}$ centimetre per second?

9. What is the measure of a velocity of 120 metres per minute when 20 centimetres per second is the unit of velocity?

10. When 3 metres per 7 seconds is the unit of velocity, what is the measure of a velocity of 126 cm. per 3 seconds?

11. What is the measure of a velocity of 40 miles per hour when the unit velocity is (1) a foot per minute, (2) a foot per second, (3) a foot per 2 minutes, (4) 2 feet per second?

12. A point has a uniform velocity of 8 feet per second. What is its displacement in 11 hours?

13. A point is moving with a uniform velocity of 20 cm. per second. What is its displacement in metres in 10 hours?

14. A point moves uniformly in a straight line at the rate of a feet per second. What is its displacement in miles in b hours?

15. A point is moving uniformly at the rate of c cm. in s seconds. How far does it go in h hours?

16. The velocity of a train is 15 miles per hour. Find (1) how many minutes it will take to go 50 yards, (2) how many seconds it will take to go 25 feet.

17. The velocity of a point is a feet per b seconds. How long does it take it to go c miles?

8. Velocity at an Instant.

An instant of time has no duration and therefore, while a point may be said to have a velocity at any given instant, an interval of time is necessary to produce a finite displacement, however small. For example, a falling body may be said to have a velocity of 10 feet per second at the instant it comes in contact with the earth, but at that instant its displacement is nothing. What is meant is, that if the body were to continue to move for one second with the velocity which it has at the instant it strikes the ground, it would have a displacement of 10 feet.

9. Average Velocity.

When a point is moving with a variable velocity, its **average velocity** for any given instant of time equals the uniform velocity of another point which has an equal total displacement in the interval.

Hence the measure of the average velocity during a given interval is obtained by dividing the measure of the distance traversed during that interval by the measure of the interval.

Thus, if s is the number of units in the total displacement, t the number of units of time in the interval, and v the measure of the average velocity, $v=\frac{s}{t}$.

If the velocity is increasing or decreasing uniformly, the average velocity for any given interval is the velocity at the middle instant of the interval. This equals half the sum of the velocities at the initial and the final instants of the interval.

If u is the measure of the velocity at the initial, and v the measure of the velocity at the final instant,

$$\text{the average velocity} = \frac{u+v}{2},$$

and if t is the measure of the interval, and s the measure of the displacement,

$$s = \left(\frac{u+v}{2}\right) t.$$

10. Measure of Variable Velocity.

If a point is moving with a varying velocity, its actual velocity at a given instant may be defined as the average velocity during an infinitely short interval containing that instant.

A variable velocity may, therefore, be approximately measured by determining the average velocity for an interval containing the instant. It is manifest that the accuracy of the determination will depend upon the length of the interval. The shorter the interval the more accurate is the result. For example, the speed of a trolley car at a certain instant cannot be accurately determined by taking the average velocity between two stopping places; but, if the space traversed by the car in a very short interval containing the instant is observed, and the average velocity for the instant calculated, the result will be approximately the velocity of the car at the instant.

Experiment.—Determination of Average Velocity.

Prepare a straight, stiff plank about three metres long. On one side fasten lengthwise two narrow strips (as in Fig 4).

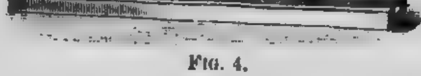

FIG. 4.

Place the plank on a table with this side upward, and with one end enough higher than the other to cause a sphere (a large glass marble answers well) to roll down the channel between the two strips readily but not too rapidly.

Adjust a metronome (Fig. 5) to tick seconds, and set the sphere free at the instant a tick is heard. Mark with a piece of chalk its position at each successive tick.

With a graduated ruler or tape determine the distance traversed by the marble during the following intervals: (*a*) 1st second, (*b*) 2nd second, (*c*) 3rd second, (*d*) 1st and 2nd seconds, (*e*) 2nd and 3rd seconds, (*f*) 1st, 2nd and 3rd seconds, etc.

1. Find the average speed during each of the foregoing intervals.

2. Carefully compare the average speeds in the first three cases.

FIG. 5.

EXERCISE II.

1. A point has displacements of 3 feet, 4 feet, 5 feet, and 6 feet in four consecutive seconds. What is its average velocity?

2. A point has displacements of 9 cm., 10 cm., 11 cm., and 12 cm. in four consecutive seconds. Find its average velocity (1) for four seconds, (2) for the first three seconds, (3) for the last three seconds.

3. A point is displaced 5 cm., 3 cm., 1 cm., -1 cm., -3 cm. in five consecutive seconds. What is its average velocity (1) for the five seconds, (2) for the first three seconds, (3) for the last three seconds, (4) for the middle three seconds?

4. A point has displacements of 2 feet, 6 feet, 10 feet, 14 feet, and 18 feet in five consecutive seconds. Show that the average velocities for the five seconds, the middle second, and the three middle seconds, are all equal.

5. A point moving with a uniformly increasing velocity has a velocity of 10 feet per second at the beginning of a certain interval of 5 seconds, and a velocity of 20 feet per second at the end of the interval. Find (1) its average velocity, (2) its displacement, during the interval.

6. A point moving with a uniformly decreasing velocity has a velocity of 60 cm. per second at the beginning of a certain interval of 10 seconds, and a velocity of 10 cm. per second at the end of the interval. Find (1) the velocity at the middle instant of the interval, (2) the displacement during the interval.

7. A point has a velocity of 10 cm. per second at the beginning, and 12 cm. per second at the end of a certain second. If its velocity is increasing uniformly, find (1) its velocity at the end of three seconds more, (2) its average velocity for the four seconds, (3) its displacement for the four seconds.

8. In one hour the velocity of a point decreases uniformly from 200 feet per second to 100 feet per second. What is the velocity at the end of each quarter of an hour, and what is the total displacement during the hour.

9. At 9 A.M. a point is moving with a velocity of 40 cm. per second, and at 1 P.M. it has a velocity of 120 cm. per second. If it moves with a uniformly increasing velocity, what velocity will it have at 11.30 A.M.?

10. At 1 P.M. a point has a velocity of 60 feet per second, and in one minute its velocity increases to 65 feet per second. If the velocity continues to increase uniformly at the same rate, find (1) its velocity at 2.15 P.M., (2) how far it goes between 1.15 and 1.20 P.M., (3) how far it goes between 2.10 and 2.15 P.M.

11. A point is moving with a uniformly decreasing motion. At the beginning of a certain second its velocity is 20 feet per second, and at the end of the same second its velocity is 18 feet per second. When will it come to rest? How far will it go during the second before it comes to rest?

12. A point is moving at a given instant with a velocity of 8 feet per second. At the end of 5 seconds its velocity is 19 feet per second. When will it have a velocity of (1) 16.8 feet per second, (2) 30 feet per second?

13. A point, starting from rest, has velocity increased uniformly 5 feet per second each second. Find (1) the velocity at the end of 10 seconds, (2) the displacement of the point during that time.

14. A point which has a velocity of 80 centimetres per second has its velocity uniformly decreased by 4 cm. per second each second. How long before it will come to rest, and how far will it move during the time?

15. The velocity of a point increases uniformly in 10 seconds from 150 cm. per second to 200 cm. per second. Find (1) the rate of increase in velocity, (2) the displacement during the 10 seconds.

16. A train moving with a velocity of 60 kilometres per hour is pulled up with a uniformly decreasing velocity in 30 seconds. What is the decrease in velocity during each second in centimetres per second?

CHAPTER II.

ACCELERATION.

1. Uniform Acceleration.

If the motion of a point is changing, the point is said to be accelerated positively or negatively, according as its velocity is increasing or diminishing.

Rate of change of velocity is called Acceleration.

The acceleration is **uniform** when equal changes of velocity take place in equal intervals of time, however short these intervals may be.

2. Measure of Uniform Acceleration.

Since acceleration is the rate of change of velocity, or the change of velocity in a unit of time, acceleration is measured by a unit of acceleration derived from the unit of velocity and the unit of time.

The unit acceleration is the acceleration of a point, the motion of which is such that its velocity is increased or diminished by the unit of velocity in each unit of time.

For example, if the centimetre is taken as the unit of length and the second as the unit of time, the **unit velocity** is **the centimetre per second,** and the **unit acceleration the centimetre per second per second.** An acceleration of 1 cm. per second per second is a change in velocity, during one second, of one centimetre per second; an acceleration of 2 cm. per second per second is a change in velocity, during one second, of two centimetres per second; and an acceleration of a cm. per

second per second, a change in velocity, during one second, of a centimetres per second.

If v is the measure of the change in velocity of a point in an interval of time, the measure of which is t, $\frac{v}{t}$ is the measure of the change of velocity in a unit of time. Then, if a is the measure of the acceleration, or rate of change of velocity,

$$a = \frac{v}{t}$$

$$\text{or} \quad v = at$$

That is, the **measure of the uniform acceleration** of a point during a given interval of time is **obtained by dividing the measure of the change of velocity during that interval by the measure of the interval,**

or

the measure of the change in velocity equals the measure of the acceleration multiplied by the measure of the interval.

EXERCISE III.

1. A point is moving with a uniform acceleration of 10 units of velocity per second. (1) What velocity will it acquire in a minute? (2) What will be the acceleration in units of velocity per minute?

2. A point is moving with a uniform acceleration of 10 feet per second per second. (1) What is the total change in velocity in a minute? (2) What is the measure of the acceleration in feet per second per minute?

3. What velocity will a body acquire in half-an-hour if the acceleration is (1) 10 centimetres per second per minute, (2) 10 centimetres per second per second?

4. A point is travelling with an acceleration of 12 feet per second per hour. Find (1) what will be its change in velocity in a minute, (2) the measure of its acceleration in feet per second per second?

5. A train acquires a velocity of 30 feet per second in one hour. If its velocity is uniformly accelerated, find (1) the velocity which it will acquire in one minute, (2) the measure of its acceleration in feet per second per second.

6. A point is travelling with an acceleration of 12 feet per second per hour. How long will it take to acquire a velocity of 2 feet per second?

7. A train, moving with a uniform acceleration, acquires a velocity of 75 feet per second in a quarter of a minute. How long will it take to acquire a velocity of 100 yards per minute?

8. A point, moving with a uniform acceleration, acquires a velocity of 60 feet per second in 10 minutes. What is the measure of its acceleration in (1) feet per second per minute, (2) yards per second per minute, (3) feet per second per second, (4) yards per second per second?

9. A point, travelling with a uniform acceleration, has its velocity increased 50 metres per second each minute. What is the measure of the acceleration in (1) metres per second per minute, (2) centimetres per second per minute, (3) metres per second per second, (4) centimetres per second per second?

10. A train, moving with a uniform acceleration, acquires an additional velocity of 60 feet per second each minute. Find (1) the measure of its acceleration in feet per second per second, (2) the measure of the velocity it acquires each minute in feet per minute, (3) the measure of the acceleration in feet per minute per minute, (4) the measure of the acceleration in feet per minute per second.

11. A point is moving with a uniform acceleration and acquires an additional velocity of 20 cm. per second each second. Find the measure of the acceleration in (1) centimetres per second per minute, (2) centimetres per minute per minute, (3) metres per minute per minute, (4) metres per minute per second, (5) metres per second per second.

12. What is the measure of an acceleration of 30 feet per second per second when the units of displacement and of time are respectively (1) the foot and the second, (2) the foot and the minute?

13. The measure of the acceleration of a falling body is 32 when the foot is the unit space and the second the unit of velocity. What

is the measure of this acceleration when the yard is the unit of space and the minute is unit of time?

14. The acceleration due to gravity of a body is 980 centimetres per second per second. What is the measure of this acceleration when the metre is the unit of displacement and the minute the unit of time?

15. Compare an acceleration of 120 feet per second per minute with an acceleration of 20 yards per minute per second.

16. How many times does an acceleration of 5 feet per second per second contain an acceleration of 18 feet per minute per minute?

17. What is the measure of an acceleration of 5 feet per second per second when 18 feet per minute per minute is the unit of acceleration?

18. Compare an acceleration of which the measure is 150 when the yard is the unit displacement and the minute the unit of time with an acceleration of which the measure is .25 when the foot is the unit of displacement and the second the unit of time.

19. What is the measure of an acceleration of 150 yards per minute per minute when an acceleration of .25 feet per second per second is the unit acceleration?

We have in the preceding exercises given a series of questions which illustrate the relations which exist among the quantities involved in problems relating to uniformly accelerated velocities. We shall now derive equations which state in a general way these relations.

Let a be the measure of the acceleration of a point moving with a uniform acceleration.

t, the measure of any interval of time in its motion.

s, the measure of the displacement for the interval.

u, the measure of the velocity of the point at the beginning of the interval.

v, the measure of the velocity of the point at the end of the interval.

Then,

$v-u$ = the change in velocity in t units of time,

$\frac{v-u}{t}$ = the change in velocity in one unit of time,
= the rate of change in velocity,
$= a$,

or,
$$\mathbf{v=u+}a\,\mathbf{t} \quad . \quad . \quad . \quad . \quad . \quad (\text{I}).$$

If the point starts from rest, $u=o$ and $v=a\,t$.

Therefore, if the acceleration remains constant,
$$v \propto t.$$

Again, since the velocity increases or decreases uniformly,

the average velocity = the velocity at the middle instant of the interval,
= half the sum of the initial and the final velocities,
$= \frac{u+v}{2}$

But the displacement = the average velocity × time (Art. 9, page 8). Therefore,
$$\mathbf{s}=\left[\frac{\mathbf{u+v}}{\mathbf{2}}\right]\mathbf{t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{II}).$$

Substituting the value for v given in (I) for v in (II), we have,
$$s=\frac{u+u+a\,t}{2}\times t$$

or,
$$\mathbf{s=u\,t+\tfrac{1}{2}}a\,\mathbf{t^2} \quad . \quad . \quad . \quad . \quad . \quad (\text{III}).$$

It should be carefully noted that when the initial velocity is in the opposite direction to the acceleration, that is when u and a are of opposite signs, the value of

s given in this equation does not always indicate the whole distance travelled by the point during the interval of time, but simply the distance between its positions at the beginning and at the end of the interval.

If the point starts from rest, $u = o$, and

$$s = \tfrac{1}{2}\, a\, t^2.$$

Therefore, if the acceleration is constant,

$$s \propto t^2.$$

From (I) $\qquad v = u + a\, t,$

or, $\qquad t = \dfrac{v - u}{a}$

From (II) $\qquad s = \left(\dfrac{v + u}{2}\right) t$

Substituting the value for t given in (I) for t in (II), we have

$$s = \frac{v+u}{2} \times \frac{v-u}{a} = \frac{v^2 - u^2}{2a}$$

or $\qquad \mathbf{v^2 = u^2 + 2}a\mathbf{s}. \qquad$ (IV).

If the point starts from rest, $u = o$, and

$$v^2 = 2\, a\, s.$$

Therefore, if the acceleration is constant,

$$s \propto v^2.$$

It will be noticed that each of the equations,

$\mathbf{v} = \mathbf{u} + a\mathbf{t}$ . . . . (I).

$\mathbf{s} = \left(\dfrac{\mathbf{u} + \mathbf{v}}{2}\right) \mathbf{t}$ . . . . (II).

$\mathbf{s} = \mathbf{u}\,\mathbf{t} + \tfrac{1}{2} a \mathbf{t}^2$ . . . . (III).

$\mathbf{v}^2 = \mathbf{u}^2 + 2 a \mathbf{s}$ . . . . (IV).

gives the relation among four of the five quantities **u**, **v**, **t**, a, and **s**, and that, if any three of these are known,

distance ... in a particular ... the second.

the values of the other two may be derived from the equations.

3. Geometrical Representations.

The results expressed in the foregoing formulas may be represented geometrically. Take, for example, the following cases:—

(1) To find the displacement of a point which has been moving with a uniform velocity of v for a time t.

Taking convenient units of length to represent time and velocity, draw a horizontal line AB to represent the time t, and a vertical line AD (Fig. 6) to represent the uniform velocity v, then the area of the rectangle ABCD, vt, will represent the displacement.

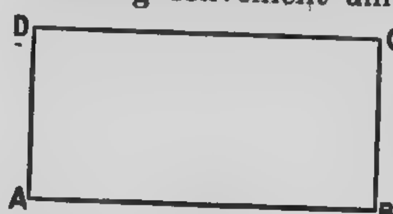

FIG. 6.

(2) To find the displacement of a point which has been moving for t seconds from rest with a uniform acceleration of a.

Taking convenient units of length to represent units of time and velocity, draw the horizontal line AB (Fig. 7) to represent the time, and the vertical line BC to represent the terminal velocity, then the area of the triangle ABC, $\frac{at \times t}{2}$, will represent the displacement, and the vertical line DE, drawn from D, the middle point of AB, will represent the average velocity.

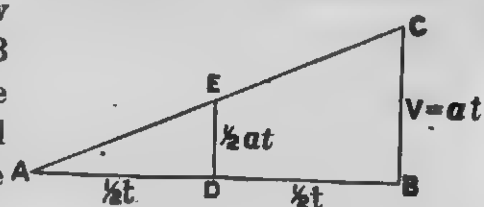

FIG. 7.

(3) Find the terminal velocity and the displacement of a point which, starting with a velocity u, has a uniform acceleration a for a time t.

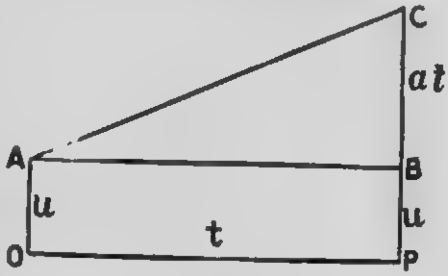

FIG. 8.

Draw a horizontal line OP (Fig. 8) to represent t and a vertical line OA to represent u. Complete the rectangle OPBA, and lay off on PB produced, BC to represent the increase in velocity during the time t. Then BC, $u+at$, will represent the terminal velocity; and the area of the figure OPCA, $ut+\frac{1}{2}at^2$, will represent the displacement.

EXERCISE IV.

1. What is the initial velocity of a point which, moving with a uniform acceleration of 10 centimetres per second per second, acquires in 10 seconds a velocity of 200 centimetres per second?

2. A body, moving at a certain instant with a velocity of 30 miles per hour, is subject to a uniform acceleration in the opposite direction, and comes to rest in 11 seconds. What was the measure of its velocity, in feet per second, 5 seconds before it stopped?

3. Find the initial velocity of a point which moves with a uniform acceleration of 20 centimetres per second per second, and acquires a velocity of 15 centimetres per second in 10 seconds. Interpret the result.

4. The velocity of a point increases uniformly in 20 seconds from 100 centimetres per second to 200 centimetres per second. Find (1) the measure of the acceleration in centimetres per second per second, (2) the velocity 3 seconds after it was 150 centimetres per second, (3) when the body was at rest.

5. A point, which has an acceleration of 32 feet per second per second, is moving with a velocity of 10 feet per second. At the same

place and at the same time another point, which has an acceleration of 16 feet per second per second, is moving in the same direction with a velocity of 170 feet per second. Find (1) when the two points will have equal velocities, (2) when the velocity of the second will be double that of the first.

6. A body, moving with a velocity of 5 centimetres per second, has a constant acceleration of 10 centimetres per second per second, in the direction of its motion. Find (1) how far it will go in 10 seconds, (2) how long it will take to go 10 centimetres.

7. A body starts with a velocity of 15 centimetres per second, and has a constant acceleration of 10 centimetres per second per second in the opposite direction. When and where will it come to rest?

8. A body, starting from rest, moves with a uniform acceleration of 20 feet per second per second. Find (1) how far the body goes in 4 seconds, (2) how far it goes in 5 seconds, (3) how far it goes in the 5th second.

9. A body starts with a velocity of 6 feet per second and has a uniform acceleration of 3 feet per second per second in the direction of its motion. At the end of 4 seconds the acceleration ceases. How far does the body move in 10 seconds from the beginning of its motion?

10. With what uniform acceleration does a point, starting from rest, describe 640 feet in 8 seconds?

11. A point, starting from rest and moving with a uniform acceleration, has a displacement of 66 feet in the 6th second. What is the measure of the acceleration in feet per second per second, and what is its displacement in the 7th second?

12. A body, moving with a uniform acceleration, has displacements in the 4th and the 6th seconds from starting of 38 feet and 54 feet respectively. Find (1) the acceleration, (2) the initial velocity, (3) the displacement in the 7th second.

13. A particle, moving with a uniform acceleration, has a displacement of 30 centimetres in the first 2 seconds of its motion. The acceleration then ceases, and the displacement for the next 2 seconds is 20 centimetres. Find (1) the initial velocity, (2) the acceleration.

14. A train, having a velocity of 20 feet per second, attains a velocity of 30 miles per hour in passing over 128 feet. If the train is moving with a uniform acceleration, what is its acceleration?

15. A trolley car, moving at the rate of 24 feet per second, is stopped with a uniformly decreasing motion in a space of 9 feet. What is the acceleration of the car?

16. A particle starts with a velocity of 23 feet per second, and its velocity is uniformly decreased at the rate of 8 feet per second per second. Find how long it will take to describe a distance of 30 feet, and how much longer to come to rest?

17. The displacement of a point moving with a uniformly decreasing motion is 100 metres in 10 seconds, and 100 metres in the next 12 seconds. In what time will it be 100 metres further on?

18. What is the velocity of a particle which, starting from rest and moving with a uniform acceleration of 8 feet per second per second, has traversed 100 feet? Find also the time required for this displacement.

19. A point which has a negative uniform acceleration of 10 centimetres per second per second is passing a certain point, and at the end of 15 centimetres farther on it has a velocity of 10 centimetres per second. Find (1) its velocity at the point, (2) in how many seconds it will return to this point.

20. A body starts from rest and moves with a uniform acceleration. If its displacement is 90 feet in the 5th second of its motion, find (1) the acceleration, (2) the velocity of the body after 10 seconds.

21. A point, moving with a uniform acceleration, describes in the last second of its motion $\frac{11}{36}$ of the whole distance. If it started from rest, how long was it in motion and through what distance did it move, if it described 4 centimetres in the 1st second?

22. If a body describes 36 feet while its velocity increases uniformly from 8 feet per second to 10 feet per second, how much further will it go before it attains a velocity of 12 feet per second?

23. A particle moves with a uniform acceleration through 80 feet in 4 seconds, and then comes to rest. Find (1) the initial velocity, (2) the average velocity during each second.

24. A point, moving with a uniform acceleration, attains a velocity of 50 centimetres per second from rest in going 125 centimetres. The acceleration is then changed to an acceleration in the opposite direction, and the point comes to rest in 250 centimetres. Compare the two accelerations.

25. A body, having a uniform acceleration, has a displacement of 27 feet in the 4th second from rest. What was the velocity at the beginning of the 4th second?

26. A particle, having a uniform acceleration, has a displacement of 32.5 cm. in the half second which elapses after the 2nd second of its motion, and a displacement in the 5th second of its motion of 110 cm. Find the initial velocity and the acceleration of the point.

27. The spaces traversed in the 1st, 2nd, 3rd, etc., seconds by a moving body are proportionately 1, 3, 5, etc. Is this consistent with the supposition that it is moving with a uniform acceleration?

28. A particle moving with a uniform acceleration of 40 centimetres per second per second, starts from a given point with a velocity of 40 centimetres per second; and three seconds afterwards another particle starts from the same point in the same direction with a velocity of 30 centimetres per second, and moves with a uniform acceleration of 60 centimetres per second per second. When and where will the second particle overtake the first?

CHAPTER III.

WORK, ENERGY AND FORCE.

I.—Work and Energy.

1. The Nature of Work and Energy.

Experiment 1.

Take the plank used in the experiment described on page 8, and two glass spheres an inch or more in diameter. Elevate one end of the plank so that if one of the spheres is very gently started to roll down the plank it will not stop, but do not elevate it enough to cause it to start from rest. Call the spheres A and B.

Start A down the plank and send B after it at a greater velocity. Observe what takes place when B overtakes A.

1. **How is B's velocity changed ?**
2. **How is A's velocity changed ?**

The above experiment illustrates in a typical way how the motion of a body may be altered. If the motion of one portion of matter is accelerated it is believed that, as in this case, the velocity of some other portion must in consequence be decreased. The body which produces the motion in the other is said "to do work" on it, while the body whose motion is accelerated is said to have work done on it. When work has been done on a body, it, in turn, acquires an increased power of doing work.

This capacity of a body to do work is called energy.

Energy must be regarded as an entity, something associated with matter in virtue of which it can do work

on other portions of matter.[1] The relation between energy and work will be better understood by a study of the following diagrams illustrative of Experiment 1.

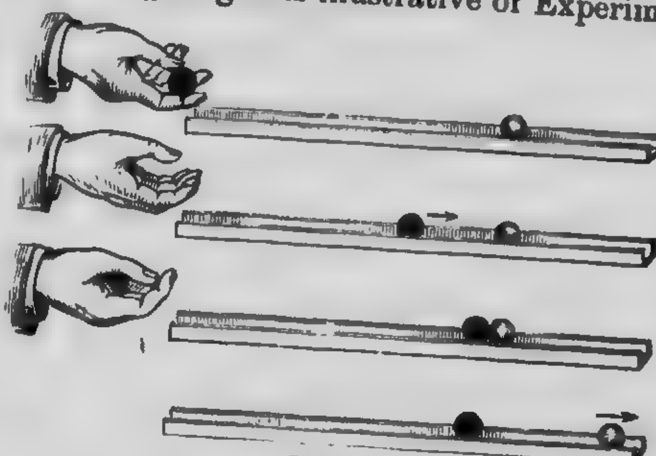

FIG. 9.

In the first, energy is going out of the hand into the first glass sphere.

In the second, energy is stored in the sphere in motion.

In the third, energy is going from the moving sphere into the one at rest.

In the fourth, the energy lost by the first sphere is stored in the second.

Work is done by the hand upon the first sphere and it, in turn, does work upon the second sphere when it comes in contact with it. In other words, **work is done whenever energy is transferred from one body to another.**

[1] "All that we know about matter relates to the series of phenomena in which energy is transferred from one portion of matter to another till in some part of the series our bodies are affected, and we become conscious of a sensation. We are acquainted with matter only as that which may have energy communicated to it from other matter. Energy, on the other hand, we know only as that which in all natural phenomena is continually passing from one portion of matter to another. It cannot exist except in connection with matter."—*Maxwell's Matter and Motion, p. 163.*

In the examples given above, work consists in the acceleration of the motion of some body, but often work is clearly done where this acceleration is not so evident.

Experiment 2.

Place a lump of lead on an anvil and strike it with a hammer. Feel the surface of the lead immediately after it has been struck by the hammer.

1. Was there an acceleration in the mass of the lead as a whole?
2. How was the temperature of the lead affected by striking it?

Experiment 3.

Attach a string to a rough heavy body, and drag it over the surface of another rough body at a uniform rate.

Is there any transference of energy while the body is being moved at a uniform rate?

It is evident that in each of the above cases work is done although there is no acceleration of the bodies as wholes, but we have reasons for thinking that the molecules of the bodies brought into contact have their motions accelerated.

But there are cases more difficult than these to explain where we have no ground for believing that the motions of even the molecules are accelerated.

If a body, say a pound weight, is lifted at a **uniform speed** vertically, work is certainly done, yet there is no **acceleration** of the body as a whole, nor have we reason to suppose that its particles are made to vibrate more rapidly. Here it is supposed that the work done on the pound weight is not stored up in the pound weight itself, but involves the acceleration of the motion of some material system not evident to our senses which is in some way influenced by the lifting of the weight.

A clock spring furnishes another example. Work is done during the process of winding it up, and the energy which is apparently transmitted to it is available for work whenever it is allowed to uncoil itself. Here, as in the case of the lifted weight, it is impossible for us to specify exactly the systems of bodies set in motion. The weight when raised and the spring when wound up, although at rest, are ready to acquire energy whenever left free to move. Since, however, the source whence they receive it is not apparent, it is customary to speak of them as if possessing the energy which they have the power of acquiring. What is apparent in such cases is that either the position of the body has been changed relatively to some other, as for example when a weight is lifted from the earth or a piece of iron is separated from a magnet; or there has been a change in the relative positions of the parts of the same body, as in the case of the clock spring when wound up or of a bow when bent.

The energy which a body is thus said to possess in virtue of its position relative to some other body, or the relative position of its parts, is called **potential** energy, while actual energy, or the energy possessed by a body in virtue of its motion, is called **kinetic** energy.

2. Upon what does the Kinetic Energy of a Body Depend?

Experiment 4.

Repeat Experiment 1, substituting in place of B a sphere C having a greater mass.

Cause C to increase the speed of A by the same amount as in the first experiment, and carefully observe the change of velocity of C.

1. In which case is the velocity of the sphere doing the work reduced the more?

2. Is the same work done on A in both cases?

3. How does the energy of A, before work is done on it, compare with its energy afterwards?

4. If the sphere doing the work has the same initial velocity in both cases, in which case has it the greater capacity for doing work?

Such experiments as the foregoing, and our ordinary observations, indicate that **a body possesses energy in virtue of its mass and its velocity.**

II.—Force.

3. Nature of Force.

In connection with the transference of energy there arises a quantity of great importance, which we shall next investigate.

We have seen in the experiments on the balls A and B (page 23), that when B does work on A the velocity of A is increased, while that of B is decreased, the increase of energy in A and the decrease of energy in B taking place during the brief interval of contact between the two bodies. While the balls are in contact we have an **action** of B on A, and a **re-action** of A on B. This action and re-action constitute what is called a **stress,** and each aspect of the stress considered by itself is called a **force.** In this case the force to which A is subject is a tendency to acceleration, while that to which B is subject is a tendency to retardation, that is, a tendency to negative acceleration. Hence either force is a **tendency to acceleration.**

These tendencies are always reciprocal, the action and the re-action being equal and opposite. The force which A is said to exert on B is equal and opposite to the force

which B exerts on A. While two bodies at least are always concerned in every force, yet it is often convenient to consider only the effect on one of the bodies without reference to the agency by which it is produced; hence force is usually defined as **any cause which tends to produce or to modify motion.**

4. Pressure, Tension.

A stress is called a **pressure** if the forces are acting towards each other, and a **tension** if they are acting away from each other.

5. Attraction, Gravity, Weight, Mass.

A body may exert a force on another with which it appears to be in no may connected. Such a force is called an **attraction.** The most familiar example is the force of **gravity,** or the attraction between a body and the earth. The measure of the force of gravity on a body is called its **weight.** The quantity of matter in a body is called its **mass.**

6. Law of Gravitation.

The tendency of bodies to move towards the earth is but a particular example of a more universal law of nature. All the evidence goes to show that each portion of matter in the universe influences the motion of every other portion. This is generally known as the **principle of universal gravitation.**

Sir Isaac Newton, from experiments and from observations of the motion of the moon, etc., arrived at the following conclusion:—

Between any two bodies in the universe there is a mutual attraction jointly proportional to the masses

f the bodies, and inversely proportional to the square n the distance between their centres of mass. For xample, if the mutual attraction between two unit masses at a distance of one foot is taken as the unit, the ttraction between three units of mass and two units of nass is :—

2×3	at a distance	of	1 foot
$\frac{2 \times 3}{2^2}$	"	"	2 feet
$\frac{2 \times 3}{3^2}$, etc.	"	"	3 feet, etc.

III.—Measure of Force.

Measure of Mass.

We cannot compare masses as we do lengths by placing hem side by side, because we are not able to judge with he eye when two bodies contain exactly an equal amount f matter. An indirect method must be used. The law f gravitation furnishes the most convenient means. ccording to this law the weights of equal masses at the ame place are equal.

The equality of the weights is usually determined by ounterpoising the masses at the ends of an equal-armed alance.

Units of Mass.

The English unit of mass is a certain quantity called a ound, the standard for which is the quantity of matter ntained in a block of platinum kept in the office of the andards in London.

The metric unit of mass is the gramme, generally now written in English, gram. It is equal to the mass of one cubic centimetre of water at four degrees centigrade. The French standard is a block of platinum which contains 1,000 grams, or one kilogram, preserved in the French archives.

9. Gravitation Units of Force.

Since the weight of a body varies as its mass when its distance from the centre of the earth remains constant, it has been found convenient to estimate the magnitudes of forces by observing the masses which they will support against gravity at the earth's surface.

In this case **we take as our unit force, the force that will support the unit mass,** *e.g.*, the pound or the gram. Thus a pound force means the force that will support the pound mass at the surface of the earth, etc.

10. Units of Mass and Units of Force Distinguished.

It will be seen that we use the word "pound" in a double sense. We use it as the name of a particular mass, and also as the name of the force required to support that mass at the surface of the earth. Whenever there is any chance of being misunderstood, it is well to use the phrase **pound mass** or **pound force,** according to which is intended. The same may be said of the words "gram," "kilogram," etc.

11. The Absolute Units of Force.

Experiment 1.

Arrange apparatus as shown in Fig. 10. The track should be about 10 feet long, as smooth as possible, and a perfect plane. The cart should be about five or six inches long and

three or four inches wide, with well-turned metal wheels two or three inches in diameter. The grooved metal pulley should be well-turned and truly mounted, and the string should be parallel with the track.

FIG. 10.

Carefully oil the wheels and the pulley. Raise one end of the track until the cart will not stop if started, but do not raise it far enough to make the cart start itself. Now load the cart with some heavy material, such as a large lump of lead, and place a much smaller mass in the scale pan.

Adjust a metronome to tick seconds, and set the cart free at the instant a tick is heard. Mark with a piece of chalk its position at each successive tick. Measure the distance the cart moves from rest in 1, 2, 3, 4, etc. seconds, and calculate the average speed during (*a*) the 1st second, (*b*) the 2nd second, (*c*) the 3rd second, (*d*) the 4th second.

It will be found that the excess of (*b*) above (*a*) is approximately the same as the excess of (*c*) above (*d*) or of (*d*) above (*c*), that is, the cart is moving with a uniform acceleration.

From the above experiment, if carefully performed, we learn that **a constant force** (in our experiment the weight of the body in the scale pan) **acting on a constant mass produces a uniform acceleration.**

Experiment 2.

Arrange once more the cart, scale pan, etc., as in the previous experiments. Load the cart with shot or sand and

place a small quantity of the same in the scale pan. Carefully ascertain the acceleration resulting. Transfer from the scale pan to the cart, until the mass supported by the string as ascertained by the use of a balance is reduced one-half, and again carefully ascertain the acceleration resulting. It will be found that the acceleration in the second case is approximately one-half the acceleration in the first case. But the force producing the acceleration in the second case is only one-half the force producing acceleration in the first case. Hence the acceleration is proportional to the force acting when the mass accelerated is constant.

From the above and similar experiments we learn that, **if different forces act on the same mass, or on equal masses, they produce accelerations directly proportional to the forces acting.**

Experiment 3.

By means of a balance prepare two masses, A and B, of some heavy material such as lead, making the mass of A double that of B. Place A and B on a book, and, holding it at a few feet above the floor, very suddenly pull the book aside, thus allowing both to drop from the same height at the same instant. Carefully observe A and B as they fall.

It will be found that they reach the ground at the same time. In other words, the acceleration of A is the same as that of B. But the force of gravity acting on A is twice that acting on B, because the mass of A is twice that of B, hence the product of the mass of A into its acceleration is also twice that of the mass of B into its acceleration.

From the foregoing experiments we see that **if two different forces act on two masses the product of the measure of the first mass into the measure of its acceleration is to the product of the measure of the second**

mass into the measure of its acceleration as the force acting on the first mass is to the force acting on the second mass, or

The magnitude of a force $\propto$ the product of the measure of the mass accelerated into the measure of the acceleration.

Hence, if P is the measure of the force, m the measure of the mass, and a the measure of the acceleration,

$$P \propto ma.$$

or $$P = kma, \text{ where } k \text{ is some constant.}$$

Now, if the unit force is taken as that force which produces in the unit mass the unit of acceleration,

that is, if $P = 1$ when $m = 1$ and $a = 1$,

the constant k must be equal to unity and

$$P = ma$$

1

Therefore, on the above conditions, **the number of units in the magnitude of any force is equal to the product of the number of units of mass in the body on which it acts into the number of units of acceleration produced in that mass by the force in question.**

The units of force thus determind are known as the **absolute,** or **kinetic,** units of force.

In the metric system, in which the units of mass, displacement, and time are respectively the gram, the centimetre, and the second, **the unit force is that force which acting on one gram mass produces an acceleration of one centimetre per second per second.**

This unit is called the **dyne.**

Hence,

$$P \text{ (in dynes)} = m \text{ (in grams)} \times a \text{ (in cm. per sec. per sec.)}.$$

In the English system the unit of force is that force which acting on one pound mass produces an acceleration of one foot per second per second. The unit is called the **poundal.**

p in poundals = m(pds) × a (in ft per sec per ...

EXERCISE V.

1. Two masses, $3m$ and $5m$, are acted on by forces which produce in their motions accelerations of 7 and 9 respectively. Compare the magnitude of the forces.

2. A force acts on a mass of m grams. Compare the acceleration with that produced by the same force acting on a mass of (1) am grams, (2) $\frac{m}{a}$ grams.

3. A force is capable of producing in a certain mass an acceleration of f cm. per sec per sec. and in another mass an acceleration of af cm. per sec. per sec. Compare the masses.

4. Two forces whose magnitudes are in the ratio 3:5 act on two bodies and communicate velocities 5 and 11 in 3 seconds. Compare the masses of the bodies.

5. Of two forces, one acts on a mass of 5 pounds and in one-eleventh of a second produces in it a velocity of 5 ft. per second, and the other acting on a mass of 625 pounds, in one minute produces in it a velocity of 18 miles per hour. Compare the forces.

6. Find the magnitude of the force expressed in dynes in each of the following cases :—

(1) The force which will produce in a mass of 20 grams an acceleration of 10 cm. per sec. per sec.

(2) The force which will produce in a mass of 5 kgm. an acceleration of 5 cm. per sec. per sec.

(3) The force which will produce in a mass of 30 grams an acceleration of 10 metres per sec. per sec.

(4) The force which will produce in a mass of 10 kgm. an acceleration of 20 cm. per min. per min.

(5) The force which acting on a mass of 3 grams for 12 seconds will impart to it a velocity of 120 cm. per sec.

(6) The force which acting on a mass of a grams for t seconds will impart to it a velocity of v cm. per sec.

7. Find the acceleration expressed in cm. per sec. per sec. in each of the following cases :—

(1) A force of 10 dynes acts on a mass of 10 grams.

(2) A force of 15 dynes acts on a mass of 5 kgm.

(3) A force of 9,800 dynes acts on a mass of 5 grams.

8. Find the mass of the body acted upon by the force in each of the following cases :—

(1) A force of 5 dynes produces in a body an acceleration of 10 cm. per sec. per sec.

(2) A force of 10 dynes acting for 5 seconds imparts to a body a velocity of 20 cm. per second.

(3) A force of 30 dynes produces in a body an acceleration of 5 metres per min. per min.

(4) A force of 1,960,000 dynes acting for 2 minutes imparts to a body a velocity of 60 cm. per sec.

9. Find the velocity acquired and the displacement in each of the following cases :—

(1) A body of mass 16 grams is acted upon by a force of 48 dynes for 5 secs.

(2) A body whose mass is 10 grams rests on a smooth horizontal plane and a force of 15 dynes acts upon it along the plane for 68 seconds.

10. During what time must a constant force of 60 dynes act upon a body whose mass is one kilogram in order to generate in it a velocity of 3 metres per second ?

CHAPTER IV.

ACCELERATION DUE TO GRAVITY.

We have learned (Chap. III) that there is in all bodies at the surface of the earth a **tendency to acceleration,** and that, if unsupported, they begin to move with a **uniform acceleration toward the earth's centre.**

1. Acceleration of a falling body independent of its mass or of the kind of matter of which it is composed.


Fig. 11.

The tendency to acceleration in a body at the earth's surface is proportional to its mass; hence all bodies, whatever their masses, fall in vacuo with the same acceleration. This may be illustrated by the following experiment.

Experiment 1.

Place a coin and a feather or small piece of paper in a "guinea and feather tube" (Fig 11), close the tube, invert it, and observe the motion of the coin and of the feather. Partially exhaust the air, invert the tube, and observe the motion of each. Now exhaust the air as completely as possible, again invert the tube, and observe the motion.

1. How does the exhausting of the air affect the relative velocities of the coin and the feather?

2. Why is the one retarded more by the air than the other?

2. Measure of the Acceleration due to Gravity.

Experiment 2.

Take two electro-magnets A and B of the form shown in Fig. 12 and connect them in one circuit with a battery C and

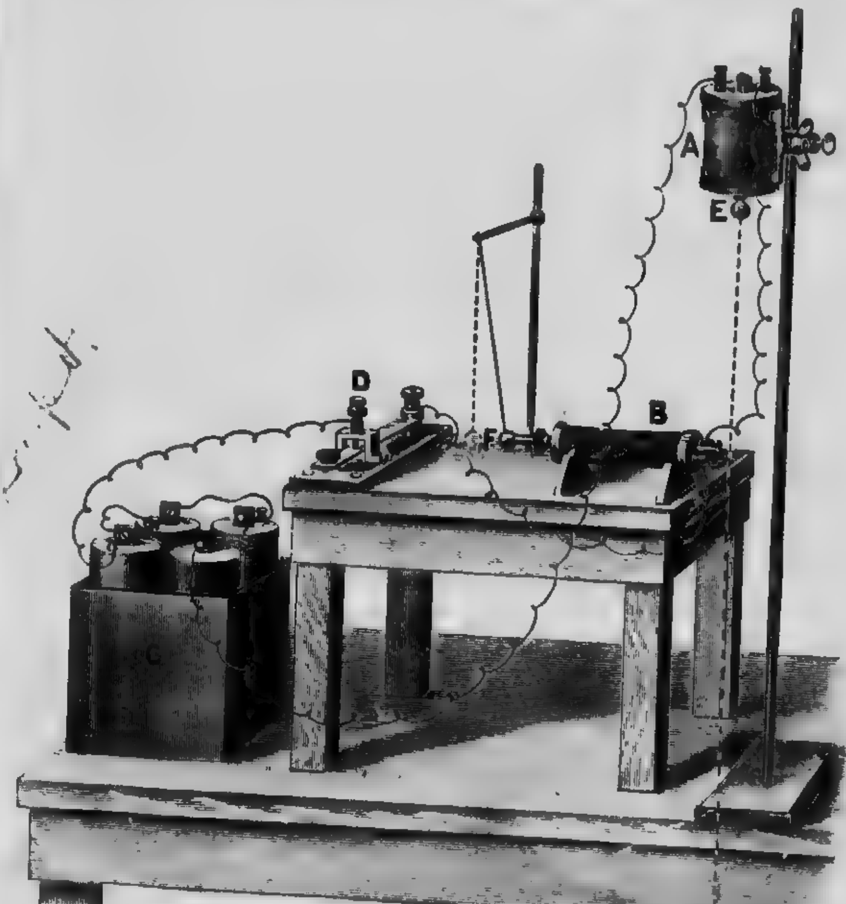

Fig. 12.

a key D. Place the electro-magnet A in such a position that it will support an iron ball E at a distance of 490 centimetres from the floor. To make certain that the ball will leave the magnet the instant that the circuit is opened

insert a piece of cardboard between the core of the magnet and the ball. By means of a thread suspend a brass pendulum bob F, which has been pierced horizontally with an iron wire, in front and slightly below the core of the electro-magnet B. The centre of the bob should be exactly 248 mm. below the point of suspension, and the iron wire should be in a line with the core of the magnet. Bring the bob up to the position shown in the figure, where it will be held by the attraction between the electro-magnet and the iron wire.

Open the circuit with the key and immediately close it again.

It will be found that the ball is heard to strike the floor at the same instant that the pendulum bob returns and strikes the core of the magnet. But a pendulum 248 mm. long makes one complete swing in one second approximately, hence a body falling freely under the action of gravity passes over 16 feet or 490 centimetres in the first second from rest.

Applying the formula,

$$s = \tfrac{1}{2}\, a\, t^2$$

we have

$$490 \text{ cm.} = \tfrac{1}{2}\, a \times 1^2,$$

or $g = 980$ centimetres per second per second, where g is the acceleration due to gravity.

3. To find the velocity of a body and the space described by it at the end of an interval of time t, when the body has been thrown vertically downward with an initial velocity of u.

Taking g as the measure of the acceleration due to gravity, and substituting in

$$v = u + a\,t \quad . \quad . \quad . \quad . \quad (1), \text{ page 17,}$$

we have

$$v = u + g\,t$$

If the body falls from rest, $u = o$, and $v = g\,t$.

To find the space described, substitute g for a in

$$s = u\,t + \tfrac{1}{2}\,a\,t^2 \quad . \quad . \quad . \quad \text{(III), page 17,}$$

therefore,

$$s = u\,t + \tfrac{1}{2}\,g\,t^2.$$

If the body falls from rest, $u = o$, and $s = \frac{1}{2}\,g\,t^2$.

4. To find the time *t* required for a body to come to rest, and the distance *s*, which it will rise when thrown vertically upward with an initial velocity of *u*.

When the body is thrown vertically upward it will lose in each unit of time g units of velocity, or in t units of time it will lose $g\,t$ units of velocity; but in the t units of time the body comes to rest, and therefore loses u units of velocity.

Therefore $\quad u = g\,t$

or $\quad t = \dfrac{u}{g}$

Again, since the initial velocity is u and terminal velocity $= o$,

The average velocity $= \dfrac{u + o}{2}$ or $\dfrac{u}{2}$

But the distance the body rises

$=$ average velocity $\times$ time,

or,

$$s = \frac{u}{2}\,t = \frac{u}{2} \times \frac{u}{g} = \frac{u^2}{2g}$$

5. To find the displacement *s* and the velocity *v* at the end of an interval of time *t*, when a body is thrown vertically upward with an initial velocity of *u*.

When the body is thrown vertically upward it will move with a negative acceleration of g. Substituting $-g$ for a in

$$v = u + a t \quad . \quad . \quad . \quad . \qquad \text{(I), page 17,}$$

we have $\quad v = u - g t.$

And substituting in

$$s = u t + \tfrac{1}{2} a t^2 \quad . \quad . \quad . \quad . \qquad \text{(III), page 17,}$$

we have $\quad s = u t - \tfrac{1}{2} g t^2.$

6. To find the time t required to reach a height h when a body is thrown vertically upward with a velocity of u. To find also the velocity v at the given height.

Substituting $-g$ for a, and h for s in

$$s = u t + \tfrac{1}{2} a t^2 \quad . \quad . \quad . \qquad \text{(III), page 17,}$$

we have $\quad h = u t - \tfrac{1}{2} g t^2$

or $\quad g t^2 - 2 u t + 2 h = o$

Therefore, $\quad t = \dfrac{u \pm \sqrt{u^2 - 2 g h}}{g}.$

Both values of t are positive. The lesser gives the time required by the body to reach the given point, and the greater the time required by it to come to rest and fall back to this point.

To find the velocity at the given height, substitute $-g$ for a, and h for s in

$$v^2 = u^2 + 2 a s \quad . \quad . \quad . \qquad \text{(IV), page 17.}$$

Therefore, $\quad v^2 = u^2 - 2 g h.$

EXERCISE VI.

[In solving the following problems the measure of the acceleration due to gravity is to be taken as 32, when the foot is the unit of length and the second is the unit of time; and as 980, when the centimetre is the unit of length and the second the unit of time.]

1. A body drops vertically from rest. What velocity will it have (1) at the end of 5 seconds, (2) when it has fallen 1,600 feet?

2. A body is thrown vertically downward with an initial velocity of 100 feet per second. Find what velocity the body will have (1) at the end of 10 seconds, (2) when it has fallen 900 feet.

3. A body is thrown vertically upward with an initial velocity of 4,900 centimetres per second. Find its velocity (1) at the end of 3 seconds, (2) when it has risen 117.6 metres.

4. A body falls from rest for 4 seconds. Find the distance fallen (1) in the four seconds, (2) in the fourth second, (3) when it has a velocity of 100 feet per second.

5. A body is thrown vertically downward with an initial velocity of 1,470 centimetres per second. Find the distance traversed in the fourth second.

6. A body is thrown vertically upward with an initial velocity of 100 feet per second. Find the height to which it will rise.

7. A body is projected vertically upward with an initial velocity of 160 feet per second. Find the distance traversed (1) in 5 seconds, (2) in the 5th second.

8. A body is thrown vertically upward with an initial velocity of 50 feet per second. What is its height when its velocity is 30 feet per second?

9. A stone is thrown vertically into the shaft of a mine with a velocity of 5.4 metres per second, and reaches the bottom in 4 seconds. What is the depth of the mine?

10. A particle is projected vertically upward with a velocity of 96 feet per second. In what time (1) will its velocity be 48 feet per second, (2) will its displacement be 144 feet?

11. A body drops vertically from rest. Find (1) when its velocity is 2,450 centimetres per second, (2) when the body is 99.225 metres from the point from which it dropped.

12. A stone is projected vertically downward with a velocity of 100 feet per second. Find (1) when its velocity is 292 feet per second, (2) when it is 900 feet from the point of projection.

13. With what velocity must a body be thrown vertically upward (1) that it may rise for 3 seconds, (2) that it may have a velocity of 30 feet per second at the end of the 3rd second, (3) that it may rise 100 feet?

14. With what velocity must a body be thrown vertically downward (1) that it may have a velocity of 100 feet per second at the end of the 2nd second, (2) that it may describe 204 feet in 3 seconds?

15. A body, thrown vertically upward, passes a point 173 feet from the point of projection with a velocity of 50 feet per second. How much further will it go, and what was the velocity with which it was projected?

16. A stone is dropped from a height of 5 metres, and at the same instant another stone is thrown vertically upward, and the stones meet half-way. Find the velocity of projection of the latter stone.

17. A particle is projected vertically upward, and it is found 34.2 that when it is just at a point 29.2 metres from the point of projection it takes 2 seconds to return to the same point. Find (1) the velocity of projection, (2) the whole height ascended.

18. A particle is projected vertically downward, and its displacement in a certain interval of time is 720 feet. In the next interval of the same length the displacement is 1,520 feet. Find the velocity of projection and the measure of the interval.

19. A balloon has been ascending with a uniform velocity for 3 seconds; and a stone let fall from it reaches the ground in 5 seconds. Find (1) the velocity of the balloon, (2) its height when the stone was let fall.

20. A stone falls from a height, and three seconds after it begins to fall another stone is thrown vertically downward from the point from which the first stone falls, with a velocity of 44.1 metres per second. When and where will the second stone pass the first?

21. A body projected vertically upward comes to rest at a point 576 feet above the point of projection. When will the body be 144 feet above the point of projection?

22. A particle is thrown vertically upward and passes a certain point with a velocity of 58.8 metres per second. How long before it will be moving downward at the same rate?

23. A stone is thrown vertically downward into the shaft of a mine with a velocity of 20 feet per second. If the stone passes through 100 feet in the last second of its fall, find the depth of the mine and the time taken by the stone in reaching the bottom.

24. A stone drops into a mine, and during the last second of its flight falls through $\frac{7}{8}$ of the total depth of the mine. What is the depth of the mine and the time taken by the stone in reaching the bottom?

25. A stone is thrown vertically upward with a velocity of 96 feet per second, and after 4 seconds from the instant of projection another stone is let fall from the same point. In what time will the first stone pass the second?

26. A body is let fall from the top of a tower 19.6 metres in height, and at the same instant another body is projected vertically upward from the base of the tower with a velocity of 19.6 metres per second. When will the bodies meet?

27. A balloon is rising with a uniform velocity of 10 feet per second when a stone is dropped from it. If the stone reaches the ground in 3 seconds, find the height of the balloon (1) when the stone dropped, (2) when the stone reached the ground.

28. A man, descending the shaft of a mine with a uniform velocity of $1\frac{2}{3}$ feet per second, drops a stone which reaches the bottom in 2 seconds. How far did the stone fall?

29. Two particles are simultaneously thrown vertically upward from the same point, the one with an initial velocity of 144 feet per second, and the other with an initial velocity of 202 feet per second. Find the height of the latter when the former reaches the ground.

30. A body after having fallen 3 seconds breaks a pane of glass and thereby loses one-third of its velocity. Find the space through which it falls in 4 seconds.

31. A stone falls freely for 3 seconds, when it pas[illegible]rough a sheet of glass, and in consequence loses one-half o[illegible]ity. If the stone reaches the ground in 2 seconds after p[illegible]rough the glass, find the height of the glass from the groun[illegible]

32. The intensity of gravity at the surface of the planet Jupiter being 2.6 times as great as it is at the surface of the earth, find approximately the time required by a body in falling from a height of 167 feet to the surface of Jupiter.

33. If a heavy body is thrown vertically up to a given height and then falls back to the earth, show that, neglecting the resistance of the air, it passes each point of its path with the same velocity when rising and when falling.

34. A stone is dropped into the shaft of a mine and is heard to strike the bottom in 12.76 seconds. If sound travels at the rate of 1,100 feet per second, what is the depth of the mine?

35. A stone dropped into a well reaches the water with a velocity of 80 feet per second, and the sound of its striking the water is heard $2\frac{7}{17}$ seconds after it is let fall. At what velocity did the sound travel?

36. A body has fallen through a feet when another body begins to fall at a point b feet below it. What is the distance which the latter body will fall before it will be passed by the former?

37. If a body is projected vertically upward with a velocity ng (where g is the measure of the acceleration due to gravity), when will its height be ng, and what will then be its velocity?

---

7. Relation between the Gravitation and the Absolute Units of Force.

When a body drops freely *in vacuo* under the action of gravity alone it moves with an acceleration of "g," therefore the measure of the force of gravity acting on a unit mass is

$$1 \times g \text{ absolute units.}$$

But the gravitation unit force is the force which will support the unit mass.

Hence, the gravitation unit force $= 1 \times g$ absolute units. Or, in the metric system,

$$\text{One gram force} = 1 \times g \text{ dynes.}$$

EXERCISE VII.

1. Express

(1) A force of 10 kgm. in dynes.

(2) A force of 10 dynes in grams force.

2. A certain force acts on a mass of 150 grams for 10 seconds, and produces in it a velocity of 50 metres per second. Compare the force with the weight of a gram.

3. A certain force acts on a mass m and generates in it an acceleration a. Find the mass which the force would statically support.

4. How long must a force of 5 units act upon a body in order to give it a momentum of 3,000 units? (The unit of momentum is that of a gram mass moving at the rate of one centimetre per second.)

5. What force acting for one minute upon a body whose mass is 50 grams will give it a momentum of 2,250 units?

6. A force of 980 dynes acts vertically upward upon a mass of 5 grams, at a place where $g=981$ cm. per sec. per sec. Find the acceleration of the body.

7. A mass of 10 kgm. is acted upon for one minute by a force which can support a mass of 125 grams. Find the momentum which it will acquire.

8. In a certain system the unit of mass is a kilogram, the unit of length is 10 cm., and the unit of time is 100 secs. Compare the unit of force with the dyne.

9. If a force acts on a body for 3 seconds from rest and generates a velocity of 60, what acceleration would this force produce in another body of double the mass?

10. If two bodies propelled from rest by equal uniform pressures describe the same space, the one in half the time that the other does, compare their final velocities and momenta.

11. A force of 64 dynes acts for 4 seconds on a mass of 2 grams initially at rest; after which the force is suddenly reversed. Find how far the mass goes in 8 seconds from rest.

12. A particle of 1 gram mass, which at a certain instant has a velocity of 96 cm. per sec., is acted on by a force of 32 dynes in a

direction opposite to the velocity. When will it be 128 cm. from its position at that instant, and what will be its velocities at those times?

13. What force must act on a mass of 48 grams to increase its velocity from 60 cm. per sec. to 90 cm. per sec. while the body passes over 120 cm.?

14. A body, acted upon by a uniform force, in 10 seconds describes a distance of 25 metres. Compare the force with the weight of the body, and find the velocity acquired.

15. A particle of 1 gram mass acted on by a constant force moves in a certain second over 20 cm., and in the next second but one it moves over 128 cm. Find the force.

16. Find the resistance (in dynes) when a body whose mass is 1.25 grams, projected along a rough table with a velocity of 48 cm. per sec., is brought to rest after 5 seconds.

17. A force which will just support a mass of 10 grams acts on a mass of 27 grams for 1 second. Find the momentum of the mass and the distance it has travelled over. At the end of the first second the force ceases to act; how far will the body travel in the next minute?

18. A mass of 20 grams is acted on by a constant force, and in the 5th and 7th seconds of its motion it passes over 108 cm. and 140 cm. respectively. Find its initial velocity and the magnitude of the force.

19. A body whose mass is 50 grams is acted on by a force for 5 seconds only. The body then describes a distance of 60 cm. in the next two seconds. Find the magnitude of the force.

20. A force which would support a mass of one kilogram acts on a body for 10 seconds, and causes it to describe 10 metres in that time. Find the mass of the body.

21. A body acted on by a constant force of 20 dynes passes over 72 cm. while its velocity increases from 16 to 20 cm. per sec. Find its mass.

22. A mass which starts from rest is acted on by a force which communicates to it in 6 seconds a velocity of 42 metres per sec. Compare the magnitude of the force with the weight of the mass.

23. A train quickens its speed uniformly from starting, and in 1 minute describes 36 metres. Compare the force exerted by the engine with the weight of the train.

24. A body whose mass is 10 grams is falling under gravity at the rate of 1,960 cm. per sec. What is the uniform force that will stop it (1) in 2 seconds, (2) in 2 centimetres ?

25. A spring-balance is carried in a balloon which is ascending vertically. Find the acceleration of the balloon when a mass of 1 kilogram hung on the spring-balance is found to indicate 1,100 grams.

26. A spring-balance is graduated at a place where $g=981$; at another place, where $g=980$, a body is tested, and the balance indicates 490 grams. What is the correct mass of the body ?

27. A mass of 10 grams falls 10 cm. from rest, and is then brought to rest by penetrating 1 cm. into some sand. Find the average pressure (in dynes) of the sand on it.

28. A man whose mass is 80 kgm. stands on an elevator. Find (in dynes) the force with which he presses the floor (1) when the elevator is going up, (2) when it is going down, with an acceleration of 490 cm. per sec. per sec.

Examples.

1 A body whose mass is m_2 grams, lies on a smooth, horizontal table (Fig. 13), and is attached by a light, inextensible string which passes over a pulley at the edge of the

FIG. 13.

table to a body whose mass is m_1 grams, which hangs freely under the action of gravity. Find (1) the acceleration, (2) the tension of the string.

Let the tension of the string be T dynes.

(1) Consider the forces acting on the mass m_1. These are

(*a*) Gravity, $m_1 g$ dynes, acting vertically downward.

(*b*) The tension of the string, T dynes, acting vertically upward.

Therefore the resultant of the forces acting on m_1 = $(m_1 g - T)$ dynes.

Let this resultant force produce in the mass m_1 an acceleration of a cm. per sec. per sec.

Then the measure of the resultant force = $m_1 a$ dynes.

Hence, $$m_1 g - T = m_1 a$$

or, $$T = m_1 g - m_1 a \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(2) Consider the forces acting on the mass m_2.

(*a*) Gravity, acting vertically downward. Since this force is balanced by the re-action of the table it will not affect the horizontal motion of the body.

(*b*) The tension of the string, acting horizontally. Since the string is *light*, it may be regarded as without mass and as exerting the same pull on m_2 as on m_1, viz., T dynes.

Also, since the string connecting the two masses is *inextensible*, the acceleration of m_2 is the same as m_1, viz., a cm. per sec. per sec.

But the force necessary to produce an acceleration of a cm. per sec. per sec. in a mass m_2 is $m_2 a$ dynes.

Therefore, $$T = m_2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

But $$T = m_1 g - m_1 a \quad . \quad . \quad . \quad (1) \text{ above.}$$

Hence, $$m_2 a = m_1 g - m_1 a$$

or $$a = \frac{m_1}{m_1 + m_2} g$$

and $$T = \frac{m_1 m_2}{m_1 + m_2} g.$$

2. Two bodies, of masses m_1 grams and m_2 grams, are connected by a light, inextensible string which passes over a small smooth pulley (Fig. 14). If m_1 is greater than m_2, determine (1) the acceleration of the system, (2) the tension of the string.

Since the pulley is *smooth* the tension of the string will be the same throughout the string. Let this tension be T dynes.

Also, since the string is inextensible, the acceleration of m_1 downward is the same as m_2 upward. Let this acceleration be a cm. per sec. per sec.

(1) Consider the forces acting on m_1. These are

(*a*) Gravity, m_1g dynes, acting vertically downward.

(*b*) The tension of the string, T dynes acting vertically upward.

Therefore, since m_1 is descending, the resultant of the forces acting on m_1

$= (m_1g - \mathrm{T})$ dynes.

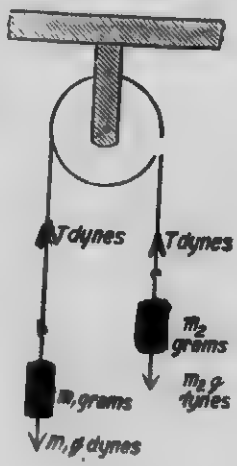

FIG. 14.

But this force is also m_1a dynes.

Hence,

$$m_1g - \mathrm{T} = m_1a$$

or,

$$\mathrm{T} = m_1g - m_1a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(2) Consider the forces acting on m_2. These are

(*a*) Gravity, m_2g dynes, acting vertically downward

(*b*) The tension of the string, T dynes, acting vertically upward.

Therefore, since m_2 is ascending, the resultant of the forces acting on m_2

$= (\mathrm{T} - m_2g)$ dynes.

But this force is also m_2a dynes.

Hence, $T - m_2g = m_2a$

or $T = m_2g + m_2a$ . . . . (2)

But $T = m_1g - m_1a$ . . . . (1) above.

Hence, $m_1g - m_1a = m_2g + m_2a$

or $a = \frac{m_1 - m_2}{m_1 + m_2} g$ cm. per sec. per cm.

and $T = m_1g - m_1 \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g.$

$= \frac{2m_1m_2}{m_1 + m_2} g$ dynes.

EXERCISE VIII.

1. A falling weight of 160 grams is connected by a string to a mass of 1,800 grams lying on a smooth flat table. Find the acceleration and the tension of the string.

2. A mass of 3 kgm. is drawn along a smooth horizontal table by a mass of 4 kgm. hanging vertically. Find the displacement in 3 seconds from rest.

3. A body of mass 9 grams is placed on a smooth table at a distance of 16 cm. from its edge, and is connected by a string passing over a pulley at the edge with a body of mass 1 gram. Find (1) the time that elapses before body reaches the edge of the table, (2) its velocity on leaving the table.

4. A mass of 10 grams hanging freely draws a mass of 60 grams along a smooth table. Find (1) the displacement in 5 seconds, (2) the displacement in the 8th second, and (3) the velocity acquired between the 7th and the 12th seconds.

5. Two masses of 100 and 120 grams are attached to the extremities of a string passing over a smooth pulley. If the value of g is 975 cm. per sec. per second, find the velocity after 8 seconds.

6. A mass of 52 grams is drawn along a table by a mass of 4 grams hanging vertically. If at the end of 4 seconds the string breaks, find the space described by each body in 4 seconds more.

7. A body falling freely acquires in one second a velocity of 981 cm. per sec. If a force equal to the weight of one gram pulls a mass of one kilogram along a smooth level surface, find the velocity when the mass has moved one metre.

8. A mass of 9 grams, descending vertically, draws up a mass of 6 grams by means of a string passing over a smooth pulley. Find the tension of the string.

9. Masses of 800 and 180 grams are connected by a string over a smooth pulley. Find the space described in (1) 5 seconds, (2) the 5th second.

10. To the ends of a light string passing over a small smooth pulley are attached masses of 977 grams and x grams. Find x so that the former mass may rise through 200 cm. in 10 seconds. ($g=981$.)

11. If bodies whose masses are m_1 and m_2 are connected by a string over a smooth pulley, find the ratio of m_1 to m_2 if the acceleration is $\frac{1}{8}$ g.

12. Two unequal masses are attached to the ends of a string passing over a smooth pulley. Find the ratio between them in order that each may pass over 490 cm. in 2 seconds, starting from rest.

13. Two masses, each equal to m grams, are connected by a string passing over a smooth pulley. What mass must be taken from one and added to the other that the system may describe 2,450 cm. in 10 seconds?

14. Masses of 400 grams and 60 grams are attached to one end of a string which passes over a smooth pulley, and a mass of 420 grams is attached to the other end of the string. After 4 seconds the 60 gram mass is detached. How long and how far will the 400 gram mass descend?

15. Prove that when two masses are suspended by a string over a smooth pulley, the tension is less than half the sum of the weights of the masses.

16. If m, the greater of two masses connected by a string over a pulley, descend with an acceleration $=a$, show that the mass which must be taken from it in order that it may ascend with the same acceleration $=\frac{4ga}{(g+a)^2}\cdot m$.

17. If two masses of 50 and 48 grams are fastened to the ends of a cord passing over a smooth pulley supported by a hook, find the pull on the hook.

18. Two equal masses of 3 grams each are connected by a light string hanging over a smooth peg; if a third mass of 3 grams be laid on one of them, by how much is the pressure on the peg increased?

19. Two masses of 520 and 480 grams are connected by a string over a smooth pulley; in 2 seconds from rest the heavier mass descends 76 centimetres. What is the acceleration due to gravity?

20. A mass of 3 grams, descending vertically, draws up a mass of 2 grams by means of a light string passing over a pulley. At the end of 5 seconds the string breaks. Find how much higher the 2 gram mass will go.

21. A string is just strong enough to support a tension equal to $\frac{1}{3}$ the sum of the weights of the masses at the extremities when the string is passed over a smooth pulley. Find the least possible acceleration that the string may not break.

22. A smooth pulley is supported by a hook, and over it passes a flexible string, to the ends of which are attached two masses of 55 and 45 grams respectively. Show that when the masses are free to move, the pull on the hook is equal to the weight of 99 grams.

CHAPTER V.

MEASUREMENT OF ENERGY, WORK, AND POWER.

1. Energy—Mass.

We have already seen (page 26) that a body possesses energy by virtue of its mass and its velocity. Let us examine the connection between the **amount of energy** a body possesses and its **mass.**

Let A and B be two bodies, the first having a mass of one pound and the second a mass of two pounds. Let both have the same velocity.

Imagine B to be divided into two equal parts.

It is manifest that the amount of energy in one part of B is the same as in the other part, or that the energy of B is double the energy of one of its equal parts. In other words, the energy of B is double that of A, or in general terms:-

The energy of a body is directly proportional to its mass.

2. Energy—Space.

Consider a clock's weight. As the weight falls work is done on the weight by the energy which causes gravitation, which energy instead of being stored up in the weight is expended as fast as it is acquired (since the weight is not accelerated) in moving the works of the clock, that is, in overcoming the friction among the wheels, etc.

Now the weight falls the same distance each unit of of time, and it is evident that the work done each unit

of time is also the same; hence the work which the energy causing gravitation does on the weight is proportional to the distance which it falls, or to generalize, **the work which the energy causing gravitation does on a falling body near the earth's surface is directly proportional to the distance the body falls.** The force brought into action in this case is the weight of the body, and therefore a constant force. Hence, we may say that **the work done in any case is directly proportional to the force brought into action, and also directly proportional to the distance through which motion takes place in the direction of the force.**

3. Gravitation, Unit of Energy, and Work.

From the foregoing it will be seen that a convenient unit of energy is the energy transferred when a unit force is brought into action and motion results through a unit distance. Thus, if a pound weight falls vertically one foot, we say that one unit of energy has been transferred. The same amount of energy is transferred, of course, if a pound force acting in any direction causes its point of application to move one foot in the direction of this force. This unit is called a **foot-pound.** We shall have, of course, a unit of energy corresponding with every combination of unit force and unit distance, for example, the **gram-centimetre,** the **kilogram-metre.** As doing work is simply transferring energy, we take as our unit of work the work done in transferring a unit of energy. The unit of work is called by the same name as the the unit of energy.

4. Absolute Units of Work.

Since the unit of energy is the energy transferred when a unit force is brought into action and motion

427 gm. ... calorie

results through a unit distance, if F is the measure of the force, s the displacement, and E the number of units of energy transferred,

$$E = Fs.$$

In the metric measurement of the absolute system, the force is measured in dynes (Art. 11, page 33), the displacement in centimetres, and the unit of energy is called the **erg.**

The erg is the energy transferred when a force of one dyne acting in any direction causes its point of application to move one centimetre in the direction of this force.

Hence,

$$E \text{ (in ergs)} = F \text{ (in dynes)} \times s \text{ (in centimetres)}.$$

As the erg is a unit so small as to require the use of large numbers in expressing ordinary quantities of work, units which are multiples of the erg are frequently employed. The most common of these is the **joule,** which is 10^7 ergs.

EXERCISE IX.

1. Find (in ergs) the work expended in raising a mass of 12 kgm. vertically through 8 metres.

2. A force of 10 dynes acts against a resistance, and in passing through one metre falls uniformly to zero. Find the work done by the force.

3. Find the work done in raising 1,000 litres of water from a well 10 metres deep.

4. Supposing that a man, whose weight is 100 kgm., in walking raises his whole mass a distance of 10 cm. at every step, and that the length of the step is 50 cm., find how much work he does in walking 500 metres.

5. A ladder 10 metres long rests against a vertical wall, and is inclined at an angle of 60° to it. How much work is done in ascending it by a man weighing 80 kgm.?

6. How much work is done in lifting 8 kgm. to a height of 12 metres above the surface of the moon, where g is 150 cm. per sec. per sec?

7. A Venetian blind consists of 30 wooden slips, each weighing 100 grams, and when the blind is down the slips are 5 cm. apart. Find the work done in drawing it up, assuming, for the purposes of your calculation, that when drawn up all the slips may be regarded as being raised to the level of the top slip.

8. A circular well 1.4 metres in diameter is 10 metres deep. Find the work expended in raising the material, supposing that a cubic metre of it weighs 2,500 kgm.

9. A well 20 metres deep is full of water. Find the depth of the surface of the water when one-quarter of the work required to empty the shaft has been done.

10. The cylinder of a steam engine has a diameter of 14 cm. and the piston moves through a distance of 20 cm. Find the work done per stroke if the pressure of the steam in the cylinder be constant, and equal to 5 kgm. per square centimetre.

P

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5. Measure of Kinetic Energy.

It was shown, Art. 2, page 26, that a body possesses energy in virtue of its mass and velocity. The amount of energy of bodily onward motion possessed by a body at any instant will be the amount of energy transferred in its being brought to rest by the action of a constant force.

Let P be the constant force, v the velocity, m its mass, a the acceleration which the force P would produce in

the mass m, s the space described by the body before it would come to rest, and E the energy transferred.

Then $\qquad P = ma$ . . . . Art. 11, page 33.

and $\qquad 0 = v^2 + 2(-a)s$, . . . page 17.

or $\qquad as = \frac{1}{2}v^2$.

But the energy transferred before the body comes to rest

$$= Ps, \quad . \quad . \quad . \quad . \quad \text{Art. 4, page 55.}$$

$$= mas = \frac{mv^2}{2}$$

Hence, $\qquad E = \dfrac{mv^2}{2}$

or, **the kinetic energy possessed by a body in motion is equal to the product of its mass into one-half of the square of its velocity.**

If m is measured in grams and v in centimetres per second, E is determined in ergs.

EXERCISE X.

1. A mass of 10 grams is thrown vertically upward with a velocity of 980 cm. per second. Find its kinetic energy (1) at the instant of projection, (2) at the end of one-half second, (3) at the end of 1 second, (4) at the end of 2 seconds.

2. Find the kinetic energy of a cannon-ball whose mass is 10 kgm. discharged with a velocity of 50 metres per second.

3. A stone of mass 6 kgm. falls from rest. What will be its kinetic energy at the end of 5 seconds?

4. A 100-gram bullet strikes an iron target with a velocity of 400 metres per second and falls dead. How much energy of bodily onward motion has the bullet lost?

5. A mass of 50 kgm. starts from rest under the action of a force, and some time afterwards is observed to be moving with a velocity of 10 metres per second. How many ergs of work have been done upon it?

6. A cricket ball, whose mass is 100 grams, is given by a blow a velocity of 20 metres per second. What is the measure of the work done?

7. Calculate the kinetic energy possessed by a stone whose mass is 1 kgm. after it has fallen from rest through a space of 1 metre.

8. If two bodies, moving with the same velocity, possess between them e units of energy, show that if their masses be m and m_1, the number of units of energy possessed by m is $\frac{me}{m+m_1}$.

9. Find the energy required to project a golf ball whose mass is 10 grams a distance of 100 metres vertically upwards.

10. A body, of mass 65, is moving with a velocity 91, the units of mass, length and time being the pound, the foot, and second respectively. Express its momentum and its kinetic energy when the units are the gram (=.035 oz.), the centimetre (=.39 in.), and the second.

11. Equal forces act for the same time upon unequal masses M and m; what is the relation between (1) the momenta generated by the forces, (2) the amounts of work done by them?

12. The kinetic energy of a raindrop is increased fourfold, while its momentum has increased threefold; in what ratios have its velocity and its mass increased?

13. A shot travelling at the rate of 200 cm. per second is just able to pierce a plank 4 cm. thick. What velocity is required to pierce a plank 12 cm. thick, assuming the resistance proportional to the thickness of the plank?

14. If a bullet moving with a velocity of 150 metres per second can penetrate 2 cm. into a block of wood, through what distance would it penetrate when moving at the rate of 450 metres per second?

15. A shot travelling at the rate of 300 metres per second can just penetrate a plank 3 cm. thick; it is forced through a plank 5 cm. thick, with a velocity of 600 metres per second. Find the velocity with which it emerges.

16. A body of mass m is moving with a velocity such that its kinetic energy is e. Show that its momentum is $\sqrt{2me}$.

17. A body is acted on by a constant force for 8 seconds. During this time 480 units of work are done upon it, and it acquires 120 units of momentum. Determine its mass, and also its velocity at the end of the given time.

18. A cube-shaped block of lead, whose edge is 10 cm. and whose sp. gr. is 11,3, falls freely from the top of a tower for 4 seconds, when it strikes a floor, breaks through, but loses, however, three-fourths of its velocity. In one-half second more it reaches the next floor and lodges there. Find (1) the momentum possessed on striking the lower floor, (2) the energy lost through the first floor.

6. Power.

Power, or activity, is the time-rate of working. The unit of power is one unit of work in one unit of time.

When the unit of work is the erg and the unit of time the second, the unit of power is the **erg-second.**

For commercial purposes, the more common unit of power is the **watt**, which is one joule-second or 10^7 erg-seconds.

For practical purposes also, the **horse-power** is frequently employed as a unit of power. It is 7.45×10^9 erg-seconds, or 745 watts.

EXERCISE XI.

1. A force of 10 dynes acting on a mass moves it through 60 cm. in 10 seconds. What is the power?

2. A force of 30 dynes acting on a mass moves it through 2 metres in a minute. What is the power?

3. A mass of 20 grams is lifted vertically a distance of 1 metre in 196 seconds. What is the rate of working?

4. On applying a dynamometer to a street car it is found that six million dynes are required to keep it in motion, while it passes over 1 kilometre in 10 minutes. Determine the rate of working in watts.

5. A force of ten million dynes is required to draw a car along a track at the rate of 36 kilometres per hour. What is the rate of working in watts?

6. A locomotive in drawing a train pulls with a force of $54 \times 10^{[illegible]}$ dynes and travels over 149 kilometres in 3 hours. What horse-power does the locomotive exert in the drawing of the train?

7. A man pumps 600 kilograms of water from a well 10 metres deep in 49 minutes. At what rate, measured in watts, is he working?

8. Calculate the horse-power of a steam engine which will raise 1,200 kilograms of water per minute from a well 149 metres deep.

9. A man whose mass is 60 kilograms walks up a hill 298 metres high in 14 minutes. What is the average power which he exerts compared with a horse-power?

10. 596,000 litres of water flow per minute over a dam 6 metres high. What is the power of the fall?

11. An engine is drawing a train whose mass is 300,000 kilograms up a smooth inclined plane of 1 in 80, at the rate of 22,350 metres per hour. What is the horse-power of the steam engine?

12. A man cycles up a hill, whose slope is 1 in 14, at the rate of 6,000 metres per hour. The mass of the man and the machine is 60 kilograms. At what rate is he working?

13. What is the horse-power of an engine which keeps a train whose mass is 60,000 kgm. moving on a horizontal track at a uniform rate of 44,700 metres per hour, the resistance due to friction, etc., being $\frac{1}{17}$ of the weight of the train?

14. Find the horse-power of an engine which can travel at the rate of 36,000 metres per hour up an incline of 1 in 70, the mass of the engine and load being 51,150 kgm., and the resistance due to friction, etc., being $\frac{1}{70}$ of the weight of the train.

15. An engine, whose horse-power is 490, pumps water from a depth of 22.35 metres. Find the number of litres raised per hour.

16. An engine of 98 horse-power, working 10 hours a day, supplies 3,000 houses with water, which it raises to a mean level of 149 metres. Find the average supply to each house.

CHAPTER VI.

COMPOSITION OF FORCES.

1. Representation of a Force.

A force is completely determined when (1) its magnitude, (2) its direction, (3) its point of application are known. These elements of a force may be completely represented by a line. The length of the line may be made to represent the magnitude of the force; the direction of the line, the direction of the force; and an extremity of the line, the point of application of the force.

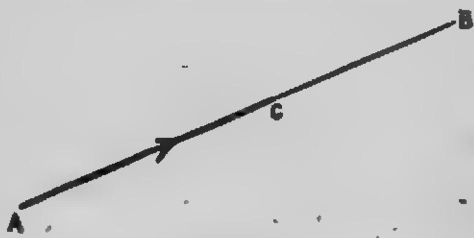


Fig. 15.

For example, if a line one centimetre in length is taken to represent a gram force, a force of 3 grams, acting at a point denoted by A, and in a direction denoted by AB, will be represented by the line AC (Fig. 15), 3 centimetres in length. An arrowhead is frequently used to indicate the direction in which the force acts.

AC represents a force acting in the direction AC, and CA a force acting in the direction C.A.

EXERCISE XII.

1. Taking a line one centimetre in length to represent a gram force, draw a line to represent a force of 12.3 grams acting (1) in a horizontal direction, (2) in a vertical direction, (3) in a direction making an angle of 45° with the horizontal.

2. Taking a line three-quarters of an inch long to represent a pound force, draw a line which represents a force of $5\frac{3}{4}$ pounds acting in a direction making an angle of 60° with the vertical.

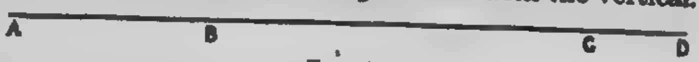


FIG. 16.

3. If AB (Fig. 16) represents a force of 60 grams, what force will be represented by (1) AC, (2) BC, (3) BD, (4) AD, (5) CD?

4. If BC (Fig. 16) represents a force of 24 pounds, what force will be represented by (1) AB, (2) AC, (3) AD, (4) BD, (5) CD?

5. If CD (Fig. 16) represents a force of 3 kilograms, what force will be represented by (1) AB, (2) AC, (3) AD, (4) BC, (5) BD?

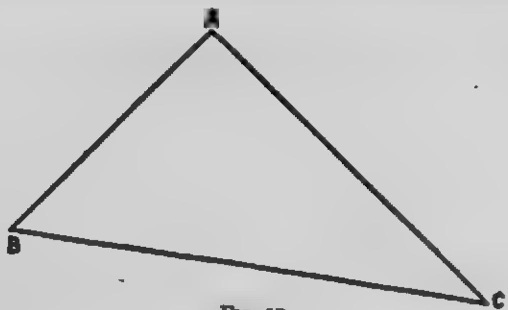


FIG. 17.

6. If 2 cm. in length represents a force of 3 grams, what are the magnitudes of the forces represented by AB, BC, CA, the sides of the triangle ABC? (Fig. 17.)

7. If AD (Fig. 18) represents a force of 2 pounds, what are the magnitudes of the forces represented by AB, AE and ED?

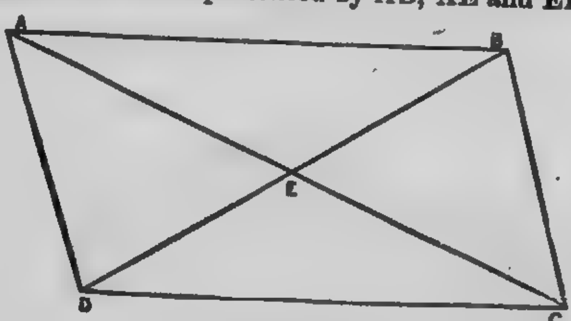


FIG. 18.

2. Resultant and Component Forces.

The single force which represents the combined effect of two or more forces is called the **resultant** of these forces, and the forces themselves are called its **components.** Forces are said to be **compounded** when two or more forces are replaced by a single force equivalent to them; that is, when the resultant is substituted for the components. A force is said to be **resolved** when a single force is replaced by two or more equivalent to it; that is, when the components are substituted for the resultant.

3. To find the resultant of two forces acting at a point when their directions are in the same straight line.

If two forces P and Q, represented by AB and BC, act in the same straight line, it is manifest that their resultant R will be represented by

$$AB + BC \qquad \text{(Fig. 19)}$$

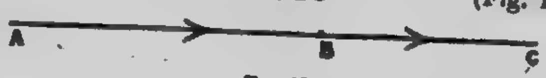


FIG. 19.

that is,

$$R = P + Q$$

when the forces act in the same direction; and by

$$AB - BC \qquad \text{(Fig. 20)}$$

FIG. 20.

that is,

$$R = P - Q$$

when the forces act in opposite directions.

4. Equilibrium.

Whenever two or more forces act upon a particle, and their individual tendencies to acceleration so counteract

each other that no motion results, the forces are said to be in **equilibrium**.

It is evident that if two forces P and Q (Fig. 21) keep a particle in equilibrium, they must be equal in magnitude and act in opposite directions in the same straight line.

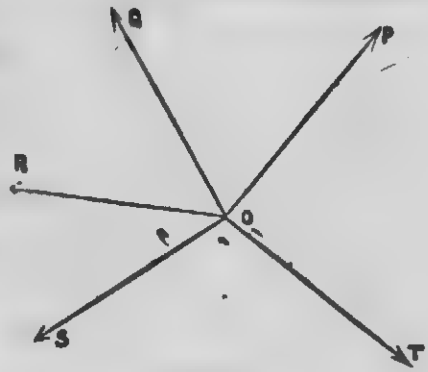


Fig. 21.

If several forces P, Q, R, S, T, in the same plane, acting at a point (Fig. 21), keep a particle in equilibrium, any one of them must equal in magnitude the resultant of all the others, and this force and the resultant of the others must act in opposite directions in the same straight line.

5. To find the resultant of two forces acting at a point when the directions of the forces are not in the same straight line—Parallelogram of Forces.

Experiment.

Prepare a smooth, flat, circular board about three feet in diameter, and rule it with cross-section ruling in inches. Mount the board on a wall with its surface in a vertical plane and with one series of lines horizontal, as shown in Fig. 22. Obtain three pulleys of the form shown in Fig. 23. Each pulley should be about three inches in diameter and mounted on a ball-bearing axle. Clamp the pulleys to the circular board at three points. Tie one end of each of three strings of suitable lengths to a small ring, and attach a ring to each of the three free ends of the strings. Pass a


Fig. 23.

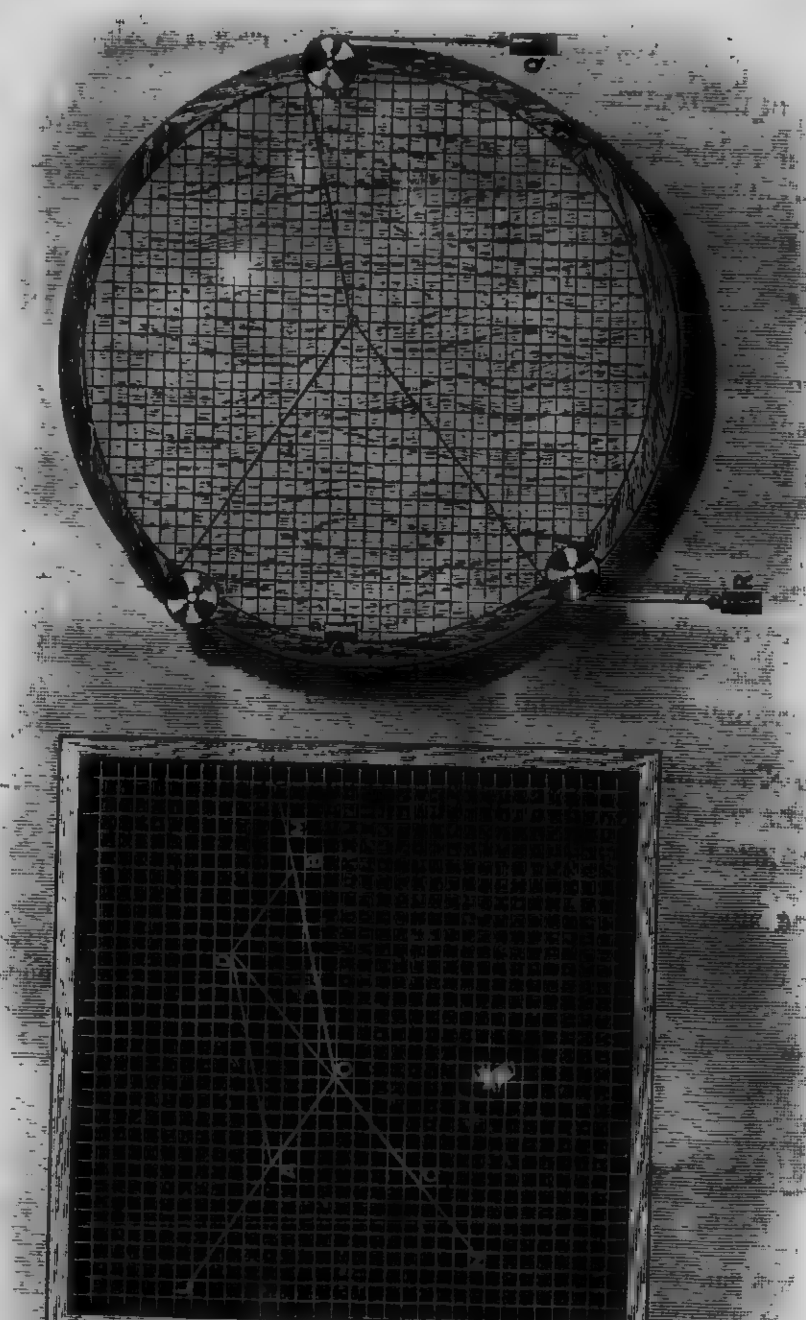


FIG. 22

string over each pulley, and suspend from the ends three such weights[1] that the combination will remain in equilibrium (Fig. 22).

The weight suspended from each string will measure the force acting along it. Let P, Q and R denote these forces.

Note the position on the board of the point of junction of the strings, and take a corresponding point O on cross-section paper or on cross-section ruling on a blackboard[2] to represent this position. Draw lines OL, OM and ON to represent the directions of the forces P, Q and R respectively. The directions of these lines can be determined by locating the positions of points L, M and N relatively to C by use of the cross-section ruling. For example, a point under the string to which P is attached, 12 divisions to the left and 8 divisions above the point of junction of the strings is noted on the circular board and a corresponding point L the same number of divisions to the left and above O is marked on the paper or blackboard. Now choosing a suitable unit of length to represent a unit of force, lay off on the line OL, OA containing P units, on the line OM, OB containing Q units, and on the line ON, OC containing R units.

By means of a ruler and a pair of compasses construct a parallelogram having OA and OB as adjacent sides.

[1] The following weights will be found convenient for this and succeeding experiments, two of each kind being supplied :—¼, ½, 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 16 ozs. They should be fitted with hooks for suspension.

[2] A good plan is to have each student in the class draw on cross-section paper, the teacher in correcting the exercises to make the drawing on permanent cross-section ruling on a blackboard placed at one side of the circular board, as shown in Fig. 22.

Draw the diagonal OD, measure its length, and determine with a straight edge whether it is in the same straight line with OC.

If the experiment is performed with care, it will be found that OD equals OC, and that it is in one and the same straight line with it.

Therefore, when OA represents the force P, and OB the force Q, OD represents a force which is equal and opposite to R; but since P, Q and R are in equilibrium, the resultant of P, and Q is equal in magnitude and opposite in direction to R. Therefore OD represents the resultant of P and Q. Hence,

When two forces acting at a point are represented in magnitude and direction by two adjacent sides of a parallelogram, the resultant of these two forces will be represented in magnitude and direction by the diagonal of the parallelogram passing through the point of junction of the two sides which represent the forces.

6. To find the resultant of two forces P and Q which act at right angles to each other.

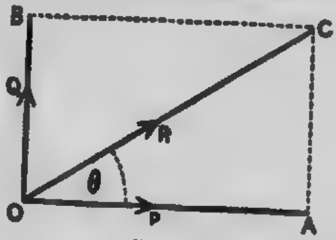


Fig. 24.

Draw the line OA to represent the force P, and the line OB to represent the force Q (Fig. 24).

Complete the parallelogram AOBC.

Then the resultant of Pand Q will be represented by OC, the diagonal of the parallelogram.

Let R denote the resultant.

Since the angle OAC is a right angle,

$$OC^2 = OA^2 + AC^2$$
$$= OA^2 + OB^2$$
$$\therefore R^2 = P^2 + Q^2$$
$$\text{or } R = \sqrt{P^2 + Q^2}.$$

If θ denotes the angle which the direction of the resultant makes with the direction of the force represented by OA,

$$\tan\theta = \frac{AC}{OA} = \frac{OB}{OA} = \frac{Q}{P}$$

EXERCISE XIII

1. Find the greatest and the least resultants of two forces whose magnitudes are 15 grams and 20 grams.

2. Find the greatest and least resultants of two forces whose magnitudes are $P + Q$ and $P - Q$.

3. Find the resultant of forces of 15 pounds and 36 pounds, acting at right angles to each other.

4. Find the resultant of two forces of 12 kilograms and 35 kilograms acting at a point, the one acting north and the other east.

5. The resultant of two forces acting at right angles is 82 pounds. If one of the forces is 80 pounds, what is the other?

6. A force of 5 P acts in a northerly direction, and the resultant of it and another force acting at the same point in an easterly direction is 13 P. What is the other force?

7. The resultant of two forces which are in the ratio 3:4 is 20 pounds when the forces act at right angles to each other. What are the forces?

8. Two forces which are in the ratio 5:12, act at right angles to each other. If the resultant of the forces is 52 grams, find the forces.

9. If two forces acting at right angles to each other are in the ratio $2:\sqrt{5}$, and their resultant is 9 pounds, find the forces.

10. Two forces acting in opposite directions to each other have a resultant of 5 pounds. If they were to act at right angles to each other, their resultant would be 25 pounds. Find the forces.

11. Two forces acting in the same direction in the same straight line have a resultant of 34 grams. When these forces act at right angles to each other their resultant is 26 grams. What are the forces?

12. Determine the resultant of the following forces acting concurrently at the same point :—12 pounds N., 24 pounds E., 7 pounds S., and 36 pounds West.

13. A weight is supported by two strings. If the strings make an angle of 90° with each other, and the tension of the one is 9 pounds, while that of the other is 12, what is the weight?

14. A boat is moored in a stream by a rope fastened to each bank. If the ropes make an angle of 90° with each other, and the force of the stream on the boat is 500 pounds, find the tention of one of the ropes if that of the other is 300 pounds.

---

7. To find the resultant of two forces P and Q inclined to each other at an angle θ.

Draw OA and OB to represent the forces P and Q respectively. Complete the parallelogram AOBC, and join OC. Then the diagonal OC will represent the resultant of P and Q.

Let R denote the resultant of P and Q.

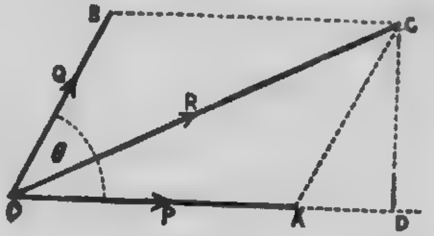


FIG. 25a.

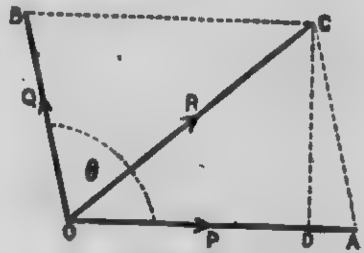


FIG. 25b.

Draw CD perpendicular to OA (Fig. 25b), or OA produced (Fig. 25a).

In Fig. 25*a*

$$\begin{aligned} OC^2 &= OA^2 + AC^2 + 2\,OA.\,AD \qquad \text{Euc. II. 12.} \\ &= OA^2 + AC^2 + 2\,OA.\,AC \cos DAC \\ &= OA^2 + OB^2 + 2\,OA.\,OB \cos\theta \\ &\quad \text{since } AC = OB \\ &\quad \text{and } DAC = \theta. \end{aligned}$$

In Fig. 25*b*,

$$\begin{aligned} OC^2 &= OA^2 + AC^2 - 2\,OA.\,AD \qquad \text{Euc. II. 13} \\ &= OA^2 + AC^2 - 2\,OA.\,AC \cos DAC \\ &= OA^2 + OB^2 + 2\,OA.\,OB \cos\theta \\ &\quad \text{since } AC = OB \\ &\quad \text{and } \cos DAC = -\cos(180^\circ - DAC) \\ &\quad\quad = -\cos\theta. \end{aligned}$$

Therefore,

$$R^2 = P^2 + Q^2 + 2PQ\cos\theta.$$

or,

$$R = \sqrt{\{P^2 + Q^2 + 2PQ\cos\theta\}}.$$

EXERCISE XIV.

1. Find the resultant of the following forces:

(1) 36 pounds and 60 pounds at an angle of 60°.

(2) 10 pounds and 10 pounds at an angle of 45°. $\frac{1}{\sqrt{2}}$

(3) 10 pounds and 10 pounds at an angle of 150°. $-\frac{\sqrt{3}}{2}$

(4) 30 pounds and 80 pounds at an angle of 120°. $-\frac{1}{2}$

(5) 2 pounds and 7 pounds at an angle of 30°. $\frac{\sqrt{3}}{2}$

(6) 2 pounds and 3 pounds at an angle of 135°. $-\frac{1}{\sqrt{2}}$

(7) 3 pounds and 16 pounds at an angle of 15°. cos $= \frac{\sqrt{6}+\sqrt{2}}{4}$

(8) 4 pounds and 11 pounds at an angle of 75°. cos $\frac{\sqrt{6}-\sqrt{2}}{4}$

(9) P acting toward the west and $P\sqrt{2}$ toward the north-east.

2. Prove that the resultant of two forces, P and P + Q, acting at an angle of 120°, is equal to the resultant of two forces, Q and P + Q, acting at the same angle.

3. Find the resultant of two forces of 10 pounds and 9 pounds acting at an angle whose tangent is $\frac{4}{3}$.

4. Find the resultant of two forces of 13 pounds and 11 pounds acting at an angle whose tangent is $\frac{12}{5}$.

5. The resultant of two forces which act at an angle of 120° is equal to one of the forces. Find the ratio of the forces.

6. The resultant of two forces which act at an angle of 60° is 13 grams. If one of the forces is 7 grams, find the other.

7. A particle is acted upon by two forces, one of which is inclined at an angle of 80° to the vertical, and the other at an angle of 40° to the vertical and on the other side of it. If one of the forces is 10 pounds, and the combined effect of the two is $2\sqrt{31}$ pounds, find the other force.

8. Two forces which act at an angle of 60° are in the ratio 3 : 5. If their resultant is 28 pounds, find the forces.

9. Two forces which are in the ratio of 1 to $\sqrt{2}$ act at an angle of 135°. If their resultant is 10 pounds, what are the forces?

10. Find the magnitude of two forces which have a resultant of $\sqrt{10}$ grams when they act at right angles to each other, and a resultant of $\sqrt{13}$ grams when they act at an angle of 60°.

11. The directions of two forces acting at a point are inclined to each other (1) at an angle of 60°, (2) at an angle of 120°, and the respective resultants are in the ratio $\sqrt{7} : \sqrt{3}$. What is the ratio of the magnitudes of the forces?

12. Two forces of two pounds each, acting at an angle of 60°, have the same resultant as two equal forces acting at right angles. What is the magnitude of these forces?

13. Six posts are placed in the ground so as to form a regular hexagon, and an elastic cord is passed around them and stretched with a force of 50 pounds. Find the magnitude and the direction of the resultant pressure on each post.

14. If one of two forces acting on a particle is 5 kilograms, and the resultant is also 5 kilograms, and at right angles to the known force, find the magnitude and the direction of the other force.

15. The magnitudes of two forces are in the ratio 3 : 5, and the direction of their resultant is at right angles to that of the smaller

force. Compare the magnitudes of the larger force and the resultant.

16. The sum of two forces is 36 pounds, and the resultant, which is at right angles to the smaller force is 24 pounds. Find the magnitude of each force.

17. Find the resultant of two forces represented by the side of an equilateral triangle and the perpendicular on this side from the opposite angle.

18. The resultant of two forces, P and Q, is $Q\sqrt{3}$, and its direction makes an angle of 30° with the direction of P. Show that P is either equal to Q or 2 Q.

19. Show that when two forces act at a point their resultant is always nearer the greater force, and the greater the angle between the forces the less is their resultant.

20. If a uniform heavy bar is supported in a horizontal position by a string slung over a peg and and attached to both ends of the bar, prove that the tension of the string will be diminished if its length is increased.

21. A weight is suspended by means of two strings of equal length attached to points in the same horizontal line. Show that if the lengths of the string are increased their tension is diminished.

22. Two forces act at a point at right angles to each other, and the magnitude of the smaller is one-half that of the resultant. Show that the angle which this force makes with the resultant is double the angle which the other force makes with it.

23. If the magnitude of one of two forces acting at a point is double that of the other, show that the angle between its direction and that of their resultant is not greater than 30°

24. If AB and AC represent two forces, and if D is the middle point of BC, show that the resultant of the forces will act along AD and will be represented in magnitude by 2 AD.

25. If D is the middle point of the side BC of a triangle ABC, show that the resultant of the forces represented by the lines AB, AC, DA is represented by the line AD.

26. The side BC of an equilateral triangle ABC is bisected at D, and AD is bisected at O. Two forces, each equal to $\sqrt{7}$ pounds, act along OB, OC. Find the magnitude and the direction of the resultant.

27. ABDC is a parallelogram, and AB is bisected in E. Show that the resultant of the forces represented by AD, AC is double of the resultant of the forces represented by AE, AC.

28. Show that the resultant of the forces represented by AC, DB, the diagonals of the parallelogram ABCD, is represented by 2 AB or 2 DC.

29. If ABC is a triangle and AB is bisected at D, AC at E, and BC at F, show that FA represents the resultant of the forces represented by BE and CD.

30. If two forces acting at a point are represented in magnitude and direction by the sides AB, BC of the triangle ABC, prove that the side AC represents the resultant.

31. Make use of the proposition stated in the last question to solve the following :—

(1) The side BC of an equilateral triangle ABC is bisected at D, and forces are represented in direction and magnitude by AB, BD. Find the magnitude of their resultant, if the force along BD is equal to a weight of 1 pound.

(2) Find the resultant of three forces represented by the sides AB, BC, CD of a rhombus ABCD.

(3) The sides AB and AC of the triangle ABC are bisected at the points D and E. Show that the resultant of the forces represented by DB, BC, CE is equivalent to the resultant of those represented by DA, AE.

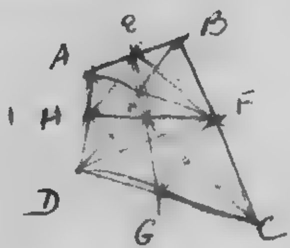
(4) If the sides BC, CA, AB, of a triangle ABC are bisected at D, E and F respectively, show that the resultant of the forces represented by AB, AC, BE will be represented by 3 FD.

32. A point is taken within or without a quadrilateral, and lines are drawn from it to the angular points of the quadrilateral.

Prove that the resultant of the forces represented by these lines is represented by the sum of the lines joining this point with the middle points of the sides of the quadrilateral.

33. A straight line is drawn parallel to the base BC of a triangle ABC, cutting AB at the point D, such that AD = 2 BD. If P is any point on the line, prove that the resultant of the forces represented by AP, BP, CP acts in this line.

34. ABCD is a quadrilateral, and AB, BC, CD, DA are bisected at the points E, F, G, H, respectively. Prove (1) that the resultant of the forces represented by AB and DC is represented by 2 HF, (2) that the resultant of the forces represented by EG and HF is represented by AC.



Res of AB . BF = AF
" " DC . CF = DF
∴ " AB . DC = AF . DF
= 2 HF

CHAPTER VII.

RESOLUTION OF FORCES.

In the preceding chapter the parallelogram of forces was employed to determine the resultant of two forces acting at a point. We shall now apply it to resolving a single force into two components which act in assigned directions.

1. To Find the Components of a Given Force in Two Given Directions.

Let R denote the given force, and α and β the angles which the components make with it.

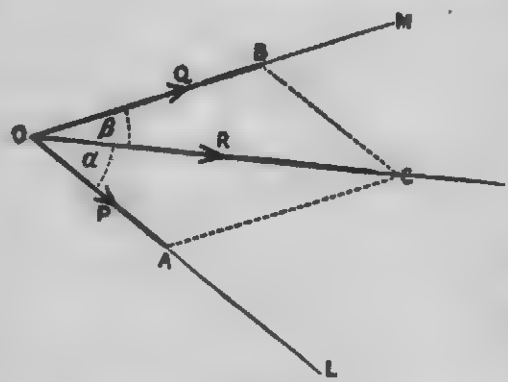


Fig. 26.

Draw the line OC (Fig. 26) to represent R; at the point O make the angle COL $= \alpha$, and the angle COM $= \beta$; and from the point C, draw CA parallel to OM, and CB parallel to OL.

Then since AOBC is a parallelogram and OC represents R, OA and OB will represent the forces P and Q respectively, where P and Q denote the required components.

2. Resolved Part.

When a force is resolved into two components at right angles to each other, each component is called the **Resolved Part** of the force in its own direction; that is,

the Resolved Part of a Force in a given direction is the force in that direction which, together with one at right angles to it, has the given force for a resultant.

The expression **resolved part** calls special attention to one of the components of a force, but it should not be forgotten that another force acts in conjunction with it, and at right angles to it. For example, if a body has a tendency to acceleration in a north-easterly direction, it has a tendency in a northerly direction accompanied by a tendency in an easterly.

3. To Find the Resolved Part of a Given Force in a Given Direction.

Draw the line OC to represent the given force in magnitude and direction (Fig. 27). From O draw the line OL

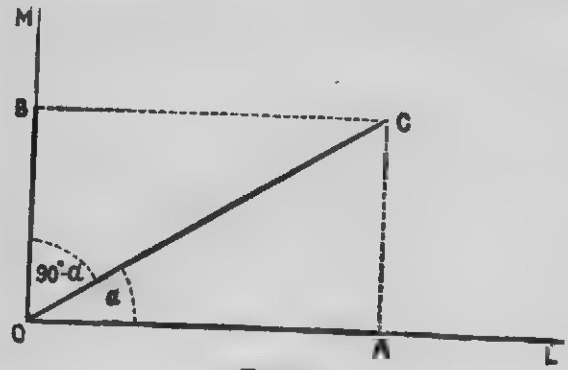


FIG. 27.

in the given direction, and the line OM at right angles to it. From C draw CA parallel to OM, and CB parallel to OL. Then the forces represented by OA and OB at right angles to each other have for their resultant the given force represented by OC.

OA, therefore, represents the resolved part of the given force in the given direction.

Let F denote the given force, and a the angle which the given direction makes with the direction of the given force.

Then, $\frac{OA}{OC} = \cos AOC = \cos a$

or, $OA = OC \cos a$,

therefore, the resolved part of a force F in a direction making an angle a with it

Hence, $= F \cos a$

The resolved part in a given direction is obtained by multiplying the given force by the cosine of the angle between the given force and the given direction.

The component perpendicular to the resolved part

$$= F \cos (90 - a)$$
$$= F \sin a$$

and the two components, therefore, are

$$F \cos a \text{ and } F \sin a.$$

EXERCISE XV.

1. Find the resolved part of a force of 10 pounds in a direction making an angle with the direction of the force of (1) 30°, (2) 45°, (3) 75°.

2. Find the horizontal and the vertical resolved parts of a force of 20 pounds, making an angle of 30° with the horizontal.

3. Find the resolved part S.W. of a force of 12 pounds S.

4. A force of 100 pounds is resolved into two equal forces at right angles to each other. What is the magnitude of either force?

5. The resultant of two forces acting at right angles is 16 pounds, and makes an angle of 30° with one of the components. Find the magnitude of the components.

6. The horizontal resolved part of a force making an angle of 30° with the horizontal is 4 pounds. Find the vertical resolved part.

7. A horse, in towing a canal boat, pulls with a force of 200 pounds. If the tow-rope is horizontal and makes an angle of 5° with the direction of the canal, find the magnitude of the force that would have to be applied in the direction of the canal to draw the boat.

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8. A horse draws a load placed upon a sleigh. If he pulls with a force of 100 pounds when the traces make an angle of 10° with the road, with what force must he pull when the traces are parallel with the road? How would a change in the inclination of the road affect the result?

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9. Show that the pressure of a perfectly smooth body resting on a perfectly smooth surface is at right angles to the surface.

10. Show that it is possible for a vessel to sail east against a south-east wind.

11. A force of 12 pounds acts along the side AB of an equilateral triangle. What is the resolved part of this force (1) along the side AC, (2) in a direction parallel to CB?

12. AB represents a force, and a circle is described on AB diameter. Show that the resolved part of this force in any direction is represented by the chord of the circle drawn in the direction through A.

---

4. To Find the Resultant of any Number of Forces Acting at a Point in given Directions Lying in one Plane.

If the forces act in the same straight line, it is evident that their resultant is the algebraic sum of the forces.

If the forces do not act in the same straight line, let O represent the given point, and through it draw two lines xx' and yy' at right angles to each other.

Let P_1, P_2, P_3 . . . denote the magnitude of the forces, and α, β, γ . . . the angles which they make with Ox.

Draw lines to represent the direction of the forces, as shown in Fig. 28.

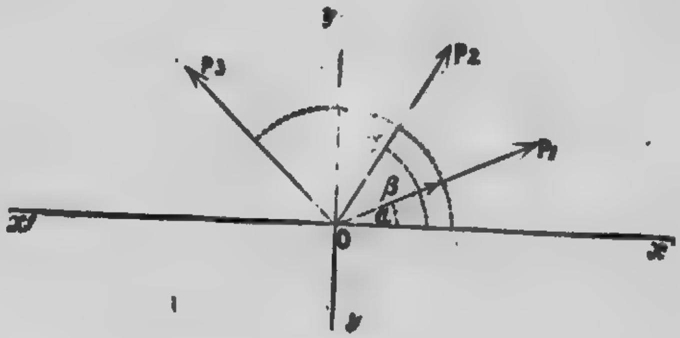


Fig. 28.

Let X denote the algebraic sum of the resolved parts of the given forces in the direction Ox,

Y denotes the algebraic sum of the resolved parts of the given forces in the direction Oy

and R denote the resultant of the given forces.

Substituting for each of the given forces its resolved parts in the directions Ox, Oy, we have (Art. 3, page 78),

$$X = P_1 \cos \alpha + P_2 \cos \beta + P_3 \cos \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$Y = P_1 \sin \alpha + P_2 \sin \beta + P_3 \sin \gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

The resultant of the given forces

= the resultant of the equivalent forces substituted for them

= the resultant of X and Y at right angles to each other.

Therefore $R = \sqrt{X^2 + Y^2}$ (Art. 6, page 69, and if θ denotes the angle which the resultant makes with Ox.

$$\tan \theta = \frac{Y}{X}$$

5. Example.

Four forces of 2 pounds, 4 pounds, $6\sqrt{3}$ pounds, and 8 pounds act at a point. If the angle between the first and second is 60°, between the second and third 90°, and between the third and fourth 150°, find the magnitude and the direction of their resultant.

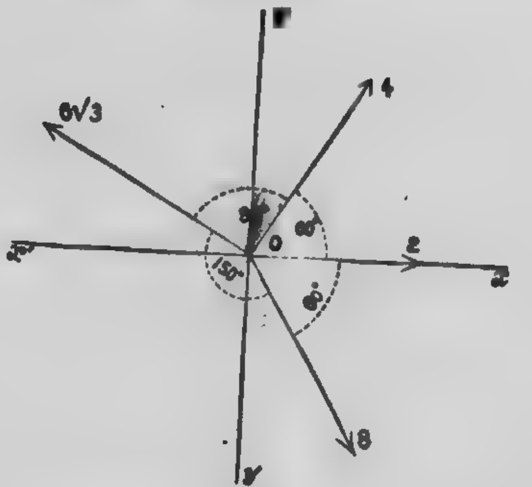


Fig. 29.

Let O be the given point, and through it draw two lines xx' and yy' at right angles to each other (Fig. 29).

Suppose the 2-pound force to act along Ox, and draw lines to represent the directions of the others, as shown in Fig. 29.

Let X = the algebraic sum of the resolved parts of the forces in the direction Ox,

and Y = the algebraic sum of the resolved parts of the forces in the direction Oy,

Substituting for each of the forces its resolved parts in the directions Ox, Oy, we have,

$$X = 2 + 4\cos 60^\circ - 6\sqrt{3}\cos 30^\circ + 8\cos 60^\circ$$
$$= 2 + 2 - 9 + 4 = -1$$
$$Y = 4\cos 30^\circ + 6\sqrt{3}\cos 60^\circ - 8\cos 30^\circ$$
$$= 2\sqrt{3} + 3\sqrt{3} - 4\sqrt{3} = \sqrt{3}$$

The resultant of the four forces

= the resultant of the equivalent forces substituted for them

= the resultant of X and Y at right angles to each other

$$=\sqrt{X^2+Y^2}$$

$$=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{4}=2 \text{ pounds.}$$

If θ denotes the angle which the resultant makes with Ox,

$$\tan\theta=\frac{Y}{X}=\frac{\sqrt{3}}{-1}=-\sqrt{3}=\tan 120^\circ$$

$$\therefore \theta=120^\circ$$

or the resultant makes an angle of 120° with the first force.

EXERCISE XVI

1. Forces of 4 pounds, 2 pounds, and 1 pound act at a point in one plane. Find the resultant when the angle between the first and second, and also between the second and third is 60°.

2. Forces of 4, 8, and $8\sqrt{3}$ pounds act at a point in one plane. If the angle between the first and the second is 60°, and the angle between the first and the third 90°, find the magnitude and the direction of the resultant.

3. Three forces, 10 pounds, 20 pounds, and 26 pounds, act at a point. If the forces act in the same plane and the angle between the directions of any two of them is 120°, find the magnitude of the resultant.

4. Four forces, 20 grams, 20 grams, 10 grams, and 10 grams, act at a point. If the forces act in one plane and the angle between the first and the second is 45°, between the second and the third 75°, and between the third and the fourth 120°, find the resultant.

5. Forces of 6 pounds, 9 pounds, and 12 pounds act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find the magnitude and the direction of the resultant.

6. Forces P, P, and Q act at a point in direction parallel to AC, CB, AB the sides of an equilateral triangle ABC. Show that their resultant is P+Q.

7. Five equal forces of 2 pounds each act along the radii of a circle which are at angular distances 30°, 60°, 90°, 120°, and 150° from a fixed radius. Find the resultant.

8. At the point O in the intersection of the diagonals of a square ABCD act forces of 2 pounds along OA, 4 pounds along OB, 3 pounds parallel to CD, and 1 pound parallel to DA. Find their resultant.

9. Forces 2 P, $\sqrt{3}$ P, 5 P, $\sqrt{3}$ P, and 2 P act at one of the angular points of a regular hexagon towards the five other angular points. Find the direction and the magnitude of the resultant.

10. Five equal forces act on a particle in directions parallel to five consecutive sides of a regular hexagon taken in order. Find the magnitude and the direction of their resultant.

---

6. Conditions of Equilibrium of any Number of Forces in the Same Plane Acting at a Point.

If a number of forces in the same plane act at a point they will be in equilibrium when the resultant is zero.

That is, when

$$R=\sqrt{X^2+Y^2}=0 \qquad \text{(Art. 6, page 69)}.$$

but the sum of the squares of two real quantities can be zero only when each quantity is separately zero.

Therefore, the forces are in equilibrium when

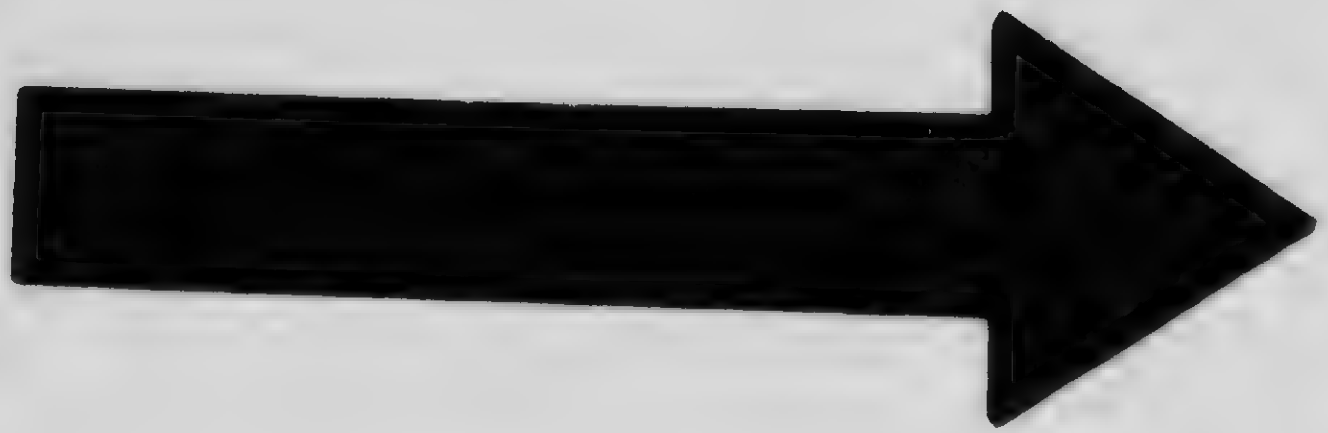
$$X=0$$
$$Y=0.$$

Hence any number of forces acting at a point are in equilibrium when the algebraic sums of the resolved parts of the forces in two directions at right angles separately vanish.

Statics

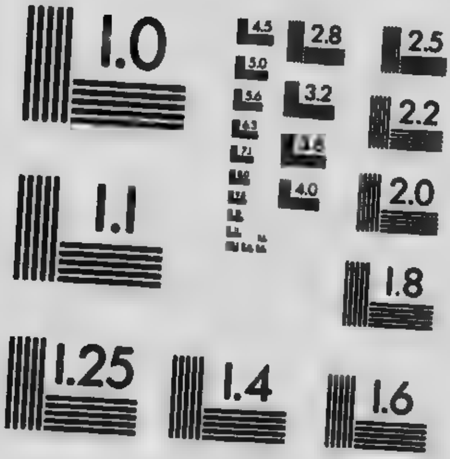
7. Examples.

1. A mass of 3 lbs. is suspended by two strings, one horizontal and the other making an angle of 30° with vertical. Find the tension of each string.



MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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Let T_1 and T_2 denote the tensions of the strings, T_1 acting along OA, and T_2 acting along OB. Draw two lines xx' and yy' at right angles to each other in the position shown in Fig. 30.

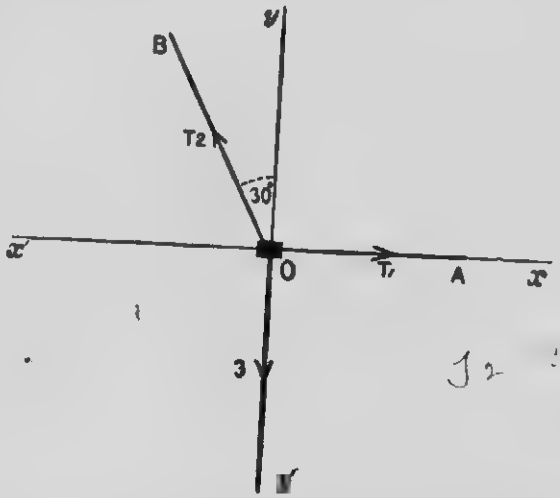


FIG. 30.

Three forces keep the mass at rest: its weight, which is 3 pounds, tension T_1, and tension T_2.

Substituting for these forces their resolved parts in the directions Ox and Oy,

$$X = T_1 - T_2 \cos 60^\circ$$
$$= T_1 - \tfrac{1}{2} T_2$$
$$Y = T_2 \cos 30^\circ - 3.$$
$$\tfrac{1}{2}\sqrt{3}\, T_2 - 3$$

But since the forces are in equilibrium

$$X = 0 \text{ and } Y = 0.$$

Hence, $T_1 - \tfrac{1}{2} T_2 = 0 \quad . \quad . \quad . \quad . \quad (1)$

$\tfrac{1}{2}\sqrt{3}\, T_2 - 3 = 0 \quad . \quad . \quad . \quad . \quad (2)$

From (2), $T_2 = \dfrac{6}{\sqrt{3}} = 2\sqrt{3}$ pounds.

Substituting for T_2 in (1)

$T_1 - \sqrt{3} = 0$ or $T_1 = \sqrt{3}$ pounds.

2. A body whose mass is 5 kilograms rests upon a smooth plane inclined at 30° to the horizon, and is acted on by four forces: (1) its weight; (2) the re-action of the plane; (3) a force equal to the weight of 2 kilograms, acting parallel to the plane and upward; and (4) a force P acting at an angle of 30° to the plane. Determine P when the body is at rest.

Let R denote the re-action of the plane. Since the plane is smooth the pressure upon it will be at right angles to it, therefore R will make a right angle with AB.

Represent the directions of the four forces by lines, as shown in Fig. 31.

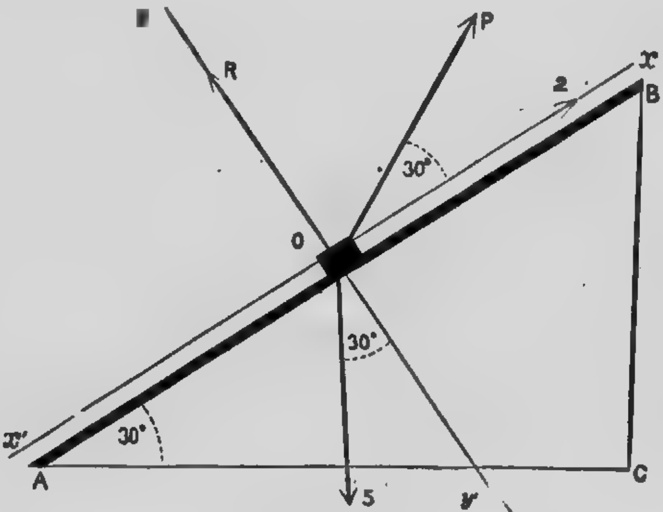


FIG. 31.

Through O draw xx' and yy' at right angles to each other.

Substituting for the forces their resolved parts in the directions Ox and Oy.

We have,

$$X = 2 + P\cos 30^\circ - 5\cos 60^\circ$$
$$= 2 + \tfrac{1}{2}\sqrt{3}\,P - \tfrac{5}{2} = \tfrac{1}{2}\sqrt{3}\,P - \tfrac{1}{2}$$
$$Y = P\cos 60^\circ + R - 5\cos 30^\circ$$
$$= \tfrac{1}{2}P + R - \tfrac{5}{2}\sqrt{3}$$

But since the forces are in equilibrium

$$X = 0 \text{ and } Y = 0$$

Hence,

$$\tfrac{1}{2}\sqrt{3}\,P - \tfrac{1}{2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\tfrac{1}{2}P + R - \tfrac{5}{2}\sqrt{3} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) $$P = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

Substituting for P in (2)

$$\frac{\sqrt{3}}{6} + R - \frac{5}{2}\sqrt{3} = 0$$

$$R = \tfrac{7}{3}\sqrt{3}$$

Hence P = the weight of a mass of $\frac{\sqrt{3}}{3}$ kilograms

and R = the weight of a mass of $\frac{7}{3}\sqrt{3}$ kilograms.

EXERCISE XVII.

1. Three forces acting at a point are in equilibrium when the angle between the first and the second is 120°, and the angle between the second and the third 150°. If the first force is 20 pounds, what is each of the others?

2. Three forces acting at a point are in equilibrium. If the angle between any two is 120°, show that the forces are equal.

3. Three forces acting at a point are in equilibrium when the angle between the first and the second is 60°, and the angle between the second and the third is 150°. Compare the forces.

4. Two forces acting on a particle are at right angles, and are balanced by a third force making an angle of 150° with one of them. If the greatest force is 10 pounds, what are the others?

5. A weight of $10\sqrt{3}$ pounds hangs at the end of a string attached to a peg. If the weight is held aside by a horizontal force, so that the string makes an angle of 30° with the vertical, find the horizontal force and the tension of the string.

6. A weight is hung at the end of a string attached to a peg. If the weight is held aside by a horizontal force, so that the string makes an angle of 60° with the vertical, compare the tension of the string and the weight.

7. A weight of 10 pounds is supported by two strings, one of which makes an angle of 30° with the vertical. If the other string makes an angle of 45° with the vertical, what is the tension of each string?

8. A string fixed at its extremities to two points in the same horizontal line supports a smooth ring weighing 2 pounds. If the two parts of the string contain an angle of 60°, what is the tension of the string?

9. A weight of 12 pounds is supported by two strings, each of which is four feet long, the ends being tied to two points in a horizontal line 4 feet apart. What is the tension of each string?

10. A picture hangs symmetrically by means of a string passing over a nail and attached to two rings fixed to the picture. What is the tension of the string, if the picture weighs 6 pounds and the angle contained by the two parts of the string is 45°?

11. A uniform bar, the weight of which is 100 pounds, is supported in a horizontal position by a string slung over a peg and attached to both ends of the bar. If the two parts of the string contain an angle of 120°, find the tension of the string.

12. A ball weighing 20 pounds slides along a perfectly smooth rod inclined at an angle of 30° with the vertical. What force applied in the direction of the rod will sustain the ball, and what is the pressure on the rod?

13. A body, the weight of which is 20 pounds, rests on a smooth plane, inclined to the horizon at an angle of 60°. Find (1) what force acting horizontally will keep the body at rest, (2) the re-action of the plane.

14. A body, the weight of which is 100 pounds, rests on a smooth plane inclined to the horizon at an angle of 30°. What force acting at an angle of 30° to the plane will keep the body at rest? What is the pressure on the plane?

15. Two weights of 2 pounds and $\sqrt{6}$ pounds respectively rest, one on each of two inclined planes which are of the same height and are placed back to back. The weights are connected by a string which passes over a smooth pulley at the common apex of

the planes. If the first plane makes an angle of 60° with the horizon, find (1) the tension of the string, (2) the pressure on each plane, (3) the inclination of the second plane to the horizon.

16. A mass of 12 grams hanging freely draws a mass of 8 grams up a smooth plane whose inclination to the horizon is 30°. Find the acceleration up the plane and the tension of the string connecting the masses.

17. A mass of 15 grams hanging freely draws a mass of 20 grams up a smooth plane whose inclination is 30°. Find the space described in the third second from rest.

18. A mass of 11 grams hanging freely draws a mass of 10 grams up a smooth inclined plane rising 3 feet in 5 feet. Find the acceleration.

19. A heavy particle slides from rest down a smooth inclined plane which is 25 cm. long and 20 cm. high. What is its velocity when it reaches the ground and how long does it take?

20. A particle slides without friction down an inclined plane, and in the 5th second after starting passes over a distance of 2,205 cm. Find the inclination of the plane to the horizon.

21. A mass m on a smooth inclined plane is connected by a string over a pulley with a mass $\frac{3}{4}m$ hanging freely. Find the inclination of the plane when m moves up a distance $\frac{1}{8}g$ in the first second.

22. A mass of 46 grams hanging freely draws a mass of 52 grams up a smooth inclined plane whose inclination is 30°. After 1 second the string breaks; how far will the 52 gram mass ascend after that?

23. A particle whose mass is 10 grams is projected up a smooth inclined plane which makes an angle of 30° with the horizon with an initial velocity of 1,960 cm. per second. Find (1) its kinetic energy at the end of 3 seconds, (2) its momentum at the end of 2 seconds, (3) when its kinetic energy will be zero.

CHAPTER VIII.

TRIANGLE AND POLYGON OF FORCES.

The conditions sufficient for equilibrium when any number of forces in one plane act at a point, are given in Art. 6, page 83.

We shall in this chapter consider another statement of these conditions.

I.—Triangle of Forces.

Experiment.

Suspend three weights, P, Q, and R by strings, as in the experiment on page 65 (Fig. 32).

Take a point O on cross-section paper or cross-section ruling on the blackboard to represent the point of junction of the strings, and draw the lines O A and A B to represent respectively the forces P and Q in magnitude and direction. The direction of A B is determined by locating on the cross-section ruling a line running from A in the same direction as that in which the string attached to Q runs from the point of junction of the strings. From B draw a line to represent in magnitude and direction the force R. The line will be found to be B O. Hence the three forces in equilibrium are represented by the three lines O A, A B, B O, the three sides of a triangle taken in order.

Repeat the experiment several times, changing the magnitudes and directions of weights P, Q and R each time.

This proposition is generally known as the **Triangle of Forces**. It may be thus stated.

1. **Triangle of Forces.**

If three forces acting at a point can be represented in magnitude and direction by the sides of a triangle taken in order, they will be in equilibrium.

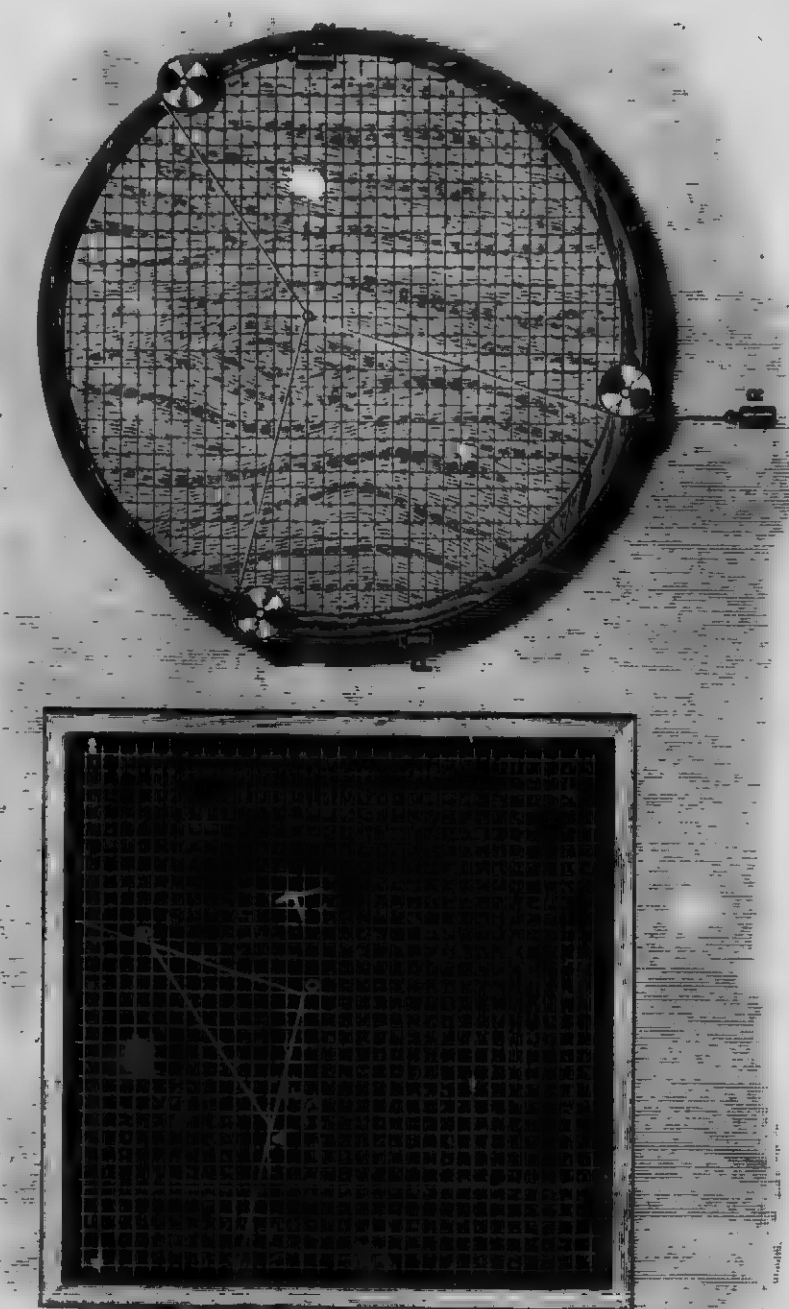


Fig. 32.

This is in reality but another statement of the parallelogram of forces, and the proposition may be derived directly from it.

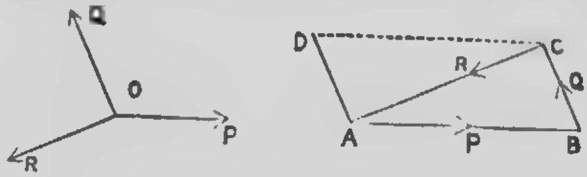


FIG. 33.

Let AB, BC, CA (Fig. 33), the sides of the triangle ABC, taken in order, represent in magnitude and direction the three forces, P, Q, R, acting at O.

Complete the parallelogram ABCD.

Since AD and BC are equal and parallel they both represent the same force.

Therefore the forces P and Q will be represented by AB, AD.

But, by the parallelogram of forces, the resultant of the forces represented by AB, AD, is represented by AC.

. Therefore the resultant of P, Q, and R, will be represented by AC, CA.

But the forces represented, AC, CA, are in equilibrium.

Hence the three forces, P, Q, and R, are in equilibrium.

The student should carefully observe

(1) That the sides of the triangle are not the lines of action of the forces, but that the forces act at a point.

(2) That the forces must be parallel to the sides of the triangle **taken in order,** P in the direction AB, Q in the direction BC, R in the direction CA, not AC.

(3) That AC represents the resultant of the forces represented by AB, BC; that is, of the forces P and Q.

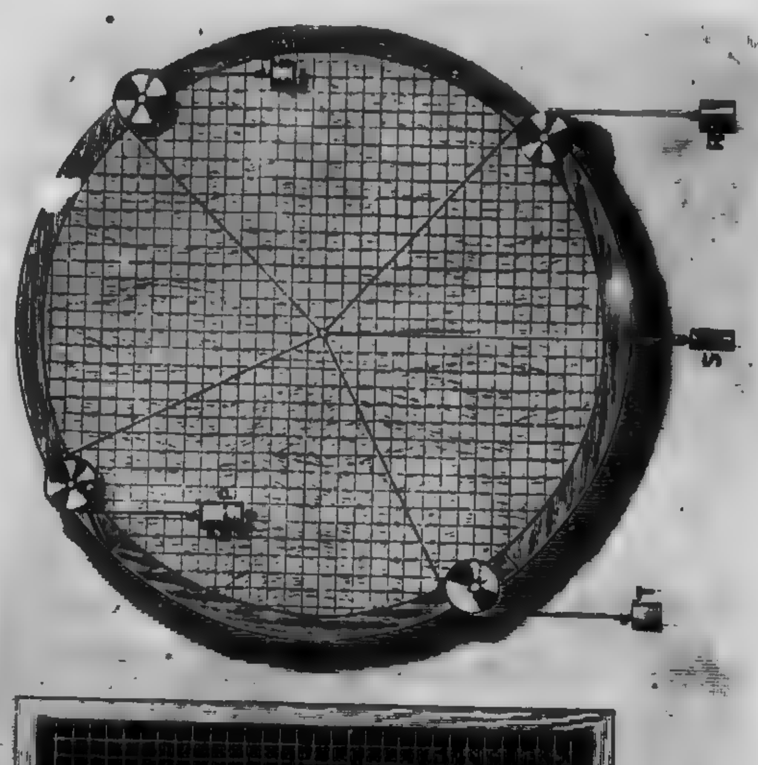


Fig. 2

2. Converse of the Triangle of Forces.

The converse of the Triangle of Forces is als. true. It may be thus stated:—

If three forces acting at a point are in equilibrium and any triangle is constructed having its sides parallel to the direction of the forces, the forces are proportional to the sides of the triangle taken in order.

II.—Polygon of Forces.

The statement of the conditions of equilibrium of three forces given in the Triangle of Forces may be extended to include any number of forces in the same plane acting at a point.

Experiment.

Arranging apparatus as in the experiment on page 65, suspend five weights as shown in Fig. 34.

Take a point O on cross-sectio . paper or cross-section ruling on a blackboard to represent the position of the point of junction of the strings, and draw the lines OA, AB, BC, and CD to represent respectively the forces P, Q, R and S in magnitude and direction. From D draw a line to represent the force T in magnitude and direction. The line will be found to be DO. Hence the five forces in equilibrium are represented by the five lines OA, AB, BC, CD, D ` the sides of a polygon taken in order.

Repeat the experiment several times, changing the number, the magnitudes and directions of the weights.

In all cases it will be found that whenever it is possible to represent the forces by the sides of a polygon taken in order the forces are in equilibrium.

The proposition is known as the **Polygon of Forces.** It may be thus stated:—

If any number of forces acting at a point can be represented in magnitude and direction by the sides of a polygon taken in order, they will be in equilibrium.

It may be derived directly from the parallelogram of forces as follows:—

Let any number of forces, P, Q, R, S, T, acting at the point O, be represented by AB, BC, CD, DE, EA, the sides, taken in order, of the polygon ABCDE (Fig. 35).

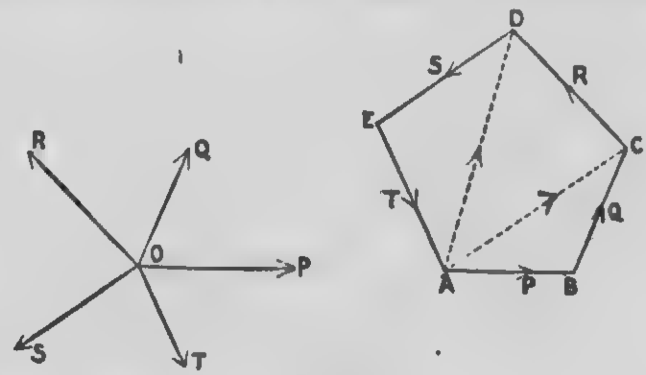


Fig. 35.

Join AC, AD.

The resultant of the forces represented by AB, BC is represented by AC, Art. (3), page 91.

Similarly the resultant of the forces represented by AC, CD is represented by AD.

And the resultant of the forces represented by AD, DE is represented by AE.

Therefore the resultant of all the forces is represented by AE, EA.

But the forces represented by AE, EA are in equilibrium.

Hence the forces are in equilibrium.

III. Examples.

1. Three forces, 6P, 7P, 8P, acting in the same plane at a point are in equilibrium. Draw lines which will represent their directions.

Construct a triangle ABC, having the side AB = 6 units of length.

BC = 7 units of length.

CA = 8 units of length. (Fig. 36).

Then the forces will be represented by the sides of the triangle ABC taken in order.

Suppose the forces to act at the point A.

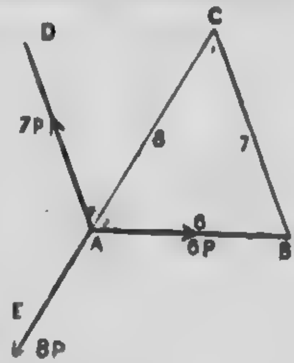


Fig. 36.

Draw AD parallel to BC, and produce CA to E.

Then AD, AB, AE, will represent the directions of the forces, because these lines are parallel to the sides of the triangle which represent the forces.

2. A pendulum, consisting of a bob weighing 4 kilograms at the end of a string one metre long, is drawn aside until the bob is 60 cm. from the vertical through the point of support, and is held in position by a horizontal string. Find the forces acting on the bob.

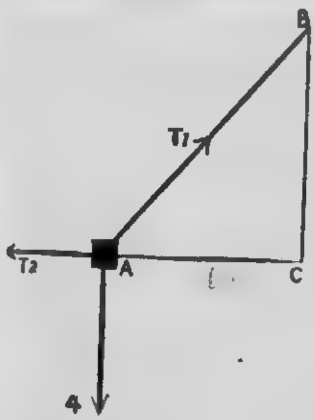


Fig. 37.

Let A represent the pendulum bob, and B the point to which the string is attached (Fig. 37). Then if T_1 denotes the tension of the pendulum string, and T_2 the tension of the horizontal string, the forces acting on the bob are T_1, T_2, and 4 kilograms, acting in the directions shown in the figure.

Draw the vertical line BC and the horizontal line AC.

Then the forces will be proportional to the sides of the triangle ABC, because the sides of the triangle are parallel to the directions of the forces.

Hence,

$$\frac{T_1}{4} = \frac{AB}{BC} = \frac{100}{80}$$

or $\quad T = 5$ kgm.

and $\quad \frac{T_2}{4} = \frac{CA}{BC} = \frac{60}{80}$

or $\quad T_2 = 3$ kgm.

3. What horizontal force is necessary to keep a mass of 100 pounds at rest on a smooth inclined plane rising 3 feet in 5? What is the re-action of the plane?

An inclined plane rising 2 feet in 5 is one in which the length AB is 5 feet when the height BC is 3 feet.

Let D represent the body (Fig. 38). If P denotes the horizontal force, and R the re-action of the plane, the body is kept in equilibrium by

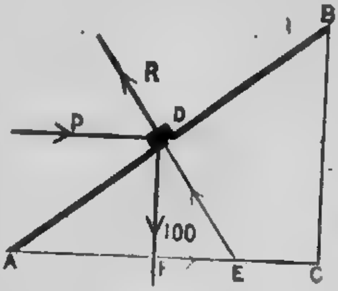


Fig. 38.

P, acting horizontally;

R, acting at right angles to AB, and 100 pounds, acting vertically downward. Produce RD to meet AC in E.

Then the forces will be proportional to the sides of the triangle EDF, because the sides of this triangle are parallel to the directions of the forces.

Hence

$$\frac{P}{100} = \frac{FE}{DF}$$

But $\quad \frac{FE}{DF} = \frac{BC}{AC}$ Since the triangle EDF is similar to the triangle BAC.

$$\therefore \quad \frac{P}{100} = \frac{3}{4}$$

or $\quad P = 75$ pounds

and $\quad \frac{R}{100} = \frac{ED}{DF} = \frac{AB}{AC} = \frac{5}{4}$

or $\quad R = 125$ pounds.

EXERCISE XVIII.

1. Can a particle be kept at rest by each of the following systems of forces acting at a point?

   (1) 4 pounds, 3 pounds, 7 pounds. no

   (2) 1 gram, 3 grams, 5 grams. no

   (3) 4 pounds, 3 pounds, 2 pounds. y

   (4) P+Q, P−Q, P, when P is greater than Q.

2. Draw lines to represent the directions of the following forces acting in one place at a point, when each system is in equilibrium:

   (1) 4 grams, 5 grams, 3 grams.

   (2) Three forces each equal to P.

   (3) 2P, P, $\sqrt{3}$P.

   (4) 5 grams, 9 grams, 4 grams.

3. Forces 5P, 12P, 13P keep a particle at rest. Show that the directions of two of the forces are at right angles to each other.

4. Find the directions in which three equal forces must act at a point to produce equilibrium.

5. Forces A+B, A−B, and $\sqrt{2(A^2+B^2)}$ keep a particle at rest. Show that the directions of two of the forces are at right angles to each other.

6. Forces of 20 pounds, 10 pounds, and 10$\sqrt{3}$ pounds act on a particle and keep it at rest. Find the angle between the directions of (1) the 20-pound force and the 10-pound force, (2) the 10-pound force and the 10$\sqrt{3}$-pound force.

7. The three forces which keep a particle at rest make angles of 60°, 150°, 150° with one another. In what proportions are the magnitudes of the forces?

8. Two forces acting at a point are at right angles and are balanced by a third force, making an angle of 150° with one of them. If the greatest force is 12 grams, what is the magnitude of each of the others?

9. A mass of 4 lbs. is suspended from a fixed point by means of a string 35 inches in length, and rests at a distance of 28 inches

from the vertical line through the fixed point when acted upon by a horizontal force. Find the horizontal force and the tension of the string.

10. A mass of 10 lbs. is suspended from a fixed point by a string 25 inches in length, and rests 20 inches below a horizontal line drawn through the fixed point when acted on by a horizontal force. Find this force and the tension of the string.

11. A string fixed at its extremities to two points in the same horizontal line supports a ring weighing 20 pounds. If the two parts of the string contain an angle of 60°, find the tension.

12. A mass of 48 lbs. is supported by two strings, which are respectively 6 feet and 8 feet long, and are fastened to two points in the same horizontal line. If the two strings are at right angles to each other, find the tension in each.

13. A mass of 65 lbs. is suspended by two strings, which are respectively 5 feet and 12 feet long, to two points in the same horizontal plane 13 feet apart. Find the tension of each string.

14. A smooth ring sustaining a mass of 48 pounds, slides along a cord fastened at two points lying in the same horizontal line 70 inches apart. If the length of the string is such that the ring rests 12 inches below the horizontal line, find the tension of the string.

15. A picture hangs symmetrically by means of a string passing over a nail and attached to two rings fixed to the picture. What is the tension of the string when the picture weighs 10 pounds, the string is 4 ft. long, and the nail 1 ft. 6 inches from the horizontal line joining the rings?

16. A body, the mass of which is 10 lbs., is suspended from a fixed point by a string 25 inches in length. What force acting at right angles to the string will hold the body in a position 20 inches below a horizontal line drawn through the point of suspension?

17. What force acting parallel to an inclined plane rising 3 feet in 5 feet will support a body the mass of which is 10 lbs.? What is the pressure on the plane?

18. What force acting horizontally will support a body the mass of which is 12 lbs., on an inclined plane rising 5 feet in 13 feet? What is the pressure on the plane?

19. What mass will be supported by a horizontal force of 9 pounds upon an inclined plane rising 3 feet in 5 feet?

20. A smooth board is fixed at an incline of 1 in 2. A mass of 1 lb. is supported on the board by a string which makes the same angle with the vertical that the board makes with the ground. What is the tension of the string?

21. ABCD is a rhombus. Show that the forces represented by AB, CB, CD and AD are in equilibrium.

22. ABCD is a parallelogram, and three forces acting at a point are represented by AC, BD and 2DA. Show that the forces are in equilibrium.

23. Three forces are represented by AB, AC, two chords of a circle drawn at right angles to each other, and DA, a diameter. Show that the forces are in equilibrium.

24. DC and AB are diameters of a circle. Three forces acting at a point are represented by AB, DC and 2BD. Show that the forces are in equilibrium.

25. Three forces are represented by the lines joining the angular points of a triangle to the middle points of the opposite sides. Show that they are in equilibrium.

26. If AB, AC represent two forces, and D is the middle point of BC, show that the two forces will be balanced by a force represented by 2DA.

27. ABCD is a parallelogram and AB is bisected in E. Show that the forces represented by AD, AC, 2AE, 2CA, are in equilibrium.

28. Four forces represented by AB, BC, CD, DE act at a point, and are balanced by a single force represented by AX. What is the position of X?

CHAPTER IX.

PARALLEL FORCES.

In Chapters VI and VII we have considered the methods of determining the resultant of any number of forces acting at a point. We have now to investigate the methods of determining the resultant of parallel forces.

Definition.

Parallel forces are said to be **like** when they act in the same direction, **unlike** when they act in opposite directions.

1. Experiment.—To Find the Resultant of Two Parallel Forces Acting Upon a Rigid Body.

Take a light uniform bar and attach to the centre *o*, and point *a*, *b*, *c*, *d*, *e*, *f*, along the rod at equal distances on each side of the centre, strings ending in rings, as shown in Fig. 39.

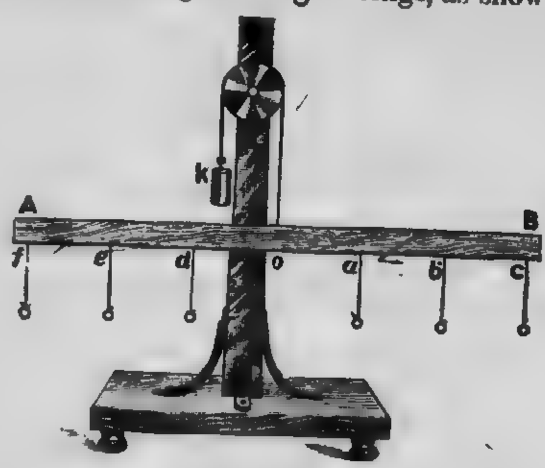


FIG. 39.

Clamp one of the pulleys used in the experiment on page 65 to an upright support and suspend the bar as shown in the

figure. Hang a weight, say 12 ozs., from *a*, and determine by trial what weights must be suspended from (a) *d* and *k* (*b*) *e* and *k*, (c) *f* and *k*, to maintain equilibrium. Repeat the experiment, suspending the first weight successively from *b* and *c*.

Now if R denotes the weight (in addition to the weight required to support the beam) suspended at *k*. and P and Q denote the weights suspended from the bar, it will be found that in each case

$$P + Q = R.$$

and

$$\frac{P}{Q} = \frac{CB}{CA},$$

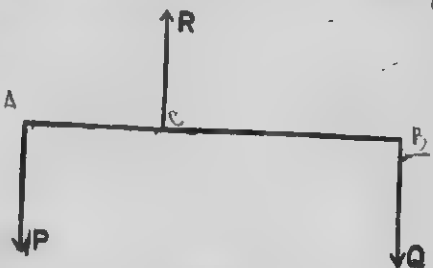


FIG. 40.

where A, B, C (Fig. 40) are respectively the points of application of P, Q, and R. But it is evident that the resultant of P and Q is a force equal and opposite to R, and that its line of action passes through C.

Hence **the magnitude of the resultant of two parallel forces acting in the same direction is the sum of the magnitudes of the components, and its point of application divides the line joining the points of application of the components inversely as the magnitudes of the forces.**

The general proposition may be deducted from the parallelogram of forces as follows:—

The following principles may be assumed:

(1) If a force act at any point of a rigid body, it may be considered to act at any other point in its line of action provided that this latter point be rigidly connected with the body. This is generally known as the **Principle of the Transmissibility of Force.**

(2) If two equal opposite forces be introduced into a system of forces acting on a body, or removed from such a system, the system of forces will not be disturbed.

Case I. When the forces are like.

Let P and Q (Fig. 41) be the forces, and A and B their points of application; let AH and BK represent them in direction and magnitude. At A and B apply two equal and opposite forces S, S, acting in the line AB. These will balance each other and will not disturb the system.

Let AD represent one force S, and BE the other force S. Complete the parallelograms AHFD and BKGE.

Let the diagonals FA and GB be produced to meet in O. Draw OC parallel to AH or BK to meet AB in C.

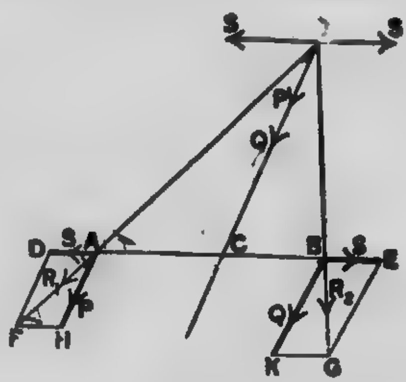


FIG. 41.

Now the forces P and S at A may be replaced by their resultant R_1, which is represented by AF, and which may be supposed to act at O.

Similarly the forces Q and S at B may be replaced by their resultant R_2, which is represented by BG, and which also may be supposed to act at O.

The force R_1 at O may be resolved into two forces, S parallel to AD, and P in the direction OC.

Also the force R_2 at O may be resolved into two forces, S parallel to BE, and Q in the direction OC.

Thus finally, instead of the two like forces P, Q applied to the rigid body at A and B respectively, we now have the four forces applied at O; namely, two equal and opposite forces each equal to S, and the two like forces P and Q acting in the line OC. The two forces each equal to S are in equilibrium and may be omitted.

Hence the resultant of the two original forces P and Q is $(P+Q)$ acting along OC, that is, acting at C in a direction parallel to that of either of the forces.

If this resultant is R, then

$$R=P+Q.$$

We have now to determine the position of the point C.

$$\frac{P}{S}=\frac{AH}{HF}. \qquad \text{(Art. 2, page 93)}$$

But the triangle AHF is similar to the triangle OCA. Consequently

$$\frac{AH}{HF}=\frac{OC}{CA} \quad \text{(Euclid VI, 4)}$$

therefore $$\frac{P}{S}=\frac{OC}{CA}. \quad \dots\dots\dots \quad (1)$$

Similarly $$\frac{Q}{S}=\frac{OC}{CB}. \quad \dots\dots\dots \quad (2)$$

Divide (1) by (2).

Hence, $$\frac{P}{Q}=\frac{CB}{CA},$$

or C divides the line AB **internally** in the inverse ratio of the forces.

Case II. When the forces are **unlike.**

Let P and Q (Fig. 42) be the forces (P being the greater), and A and B their points of application; and

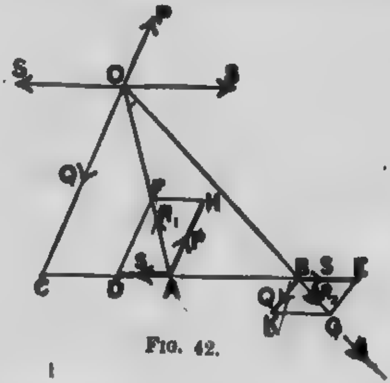


FIG. 42.

let AH and BK represent them in direction and magnitude. At A and B apply two equal and opposite forces S, S, acting in the line AB. These will balance each other and not disturb the system.

Let AD represent one force S and BE the other force S.

Complete the parallelograms AHFD and BKGE.

Let the diagonals AF and GB be produced to meet in O. Draw OC parallel to AH or BK to meet BA produced in C.

Now the forces P and S at A may be replaced by resultant R_1, which is represented by AF, and which may be supposed to act at O.

Similarly the forces Q and S at B may be replaced by their resultant R_2, which is represented by BG, and which also may be supposed to act at O.

The force R_1 at O may be resolved into two forces, S parallel to AD, and P in the direction CO produced.

Also the force R_2 at O may be resolved into two forces, S parallel to BE, and Q in the direction OC.

Thus finally, instead of the two unlike forces P and Q applied to the rigid body at A and B respectively, we now have the four forces applied at O; namely, two equal and opposite forces each equal to S, and two unlike forces P and Q acting in the line CO. The two forces each equal to S are in equilibrium and may be omitted.

Hence the resultant of the two original forces is $(P - Q)$ acting along CO, that is, acting at C in a direction parallel to that of either of the forces.

If this resultant is R, then

$$R = P - Q.$$

We have now to determine the position of the point C.

$$\frac{P}{S} = \frac{AH}{HF} \qquad \text{(Art. 2, p. 93)}$$

But the triangle AHF is similar to the triangle OCA, consequently

$$\frac{AH}{HF} = \frac{OC}{CA} \quad \text{(Euclid VI, 4.)}$$

therefore

$$\frac{P}{S} = \frac{OC}{CA} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly

$$\frac{Q}{S} = \frac{OC}{CB} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Divide (1) by (2).

Hence,

$$\frac{P}{Q} = \frac{CB}{CA}$$

or the point C divides the line HB **externally** in the inverse ratio of the forces.

Summary. When two parallel forces act on a rigid body.

1. The magnitude of the resultant is the algebraic sum of the magnitudes of the components.

2. The line of action of the resultant is parallel to the lines of action of the components; also, when the component forces are like, its direction is the same as that of the two forces, and, when the forces are unlike, its direction is the same as that of the greater component.

3. The point of application of the resultant divides the line joining the points of application of the components, internally when the forces are like, and externally when the forces are unlike, inversely as the magnitudes of the forces.

1. Couple.

If the forces P and Q (Fig. 42) be equal, $R=0$, and AF and GB being parallel, $BC=\infty$. This indicates that when two equal unlike parallel forces act on a body they cannot be replaced by any single force acting at a finite distance. Such a system of unlike parallel forces is called a **couple**.

2. Resultant of a Number of Parallel Forces.

When a number of parallel forces act on a rigid body, their resultant can be found by taking two of them and finding the magnitude and line of action of their resultant, and then combining this resultant with a third force and so on. It is evident that if R is the resultant of the parallel forces P_1, P_2, P_3, ... etc.

$$R = P_1 + P_2 + P_3 + \quad . \quad . \quad . \quad . \quad . \quad = \Sigma\{P\}.$$

If one or more of the forces act in the opposite direction its sign must, of course, be changed.

EXERCISE XIX.

1. Find the magnitude and point of application of the resultant of two like parallel forces of 3 dynes and 2 dynes acting at points 5 metres apart.

2. Find the magnitude and point of application of the resultant of two unlike parallel forces of 17 dynes and 25 dynes acting at points 8 metres apart.

3. The resultant of two parallel forces is 15 pounds, and acts at a distance of 4 feet from one of them whose magnitude is 7 pounds. Find the position and magnitude of the second force, when (1) the forces are like, (2) when unlike.

4. Two men, of the same height, carry on their shoulders a pole 6 feet long, and a mass of 121 pounds is slung on it, 30 inches from one of the men. What portion of the weight does each man support?

5. Two men support a weight of 112 pounds on a weightless pole which rests on the shoulder of each. The weight is twice as far from the one as from the other. Find what weight each supports.

6. A man carries two buckets of water by means of a pole which he holds in his hand at a point three-fifths of its length from one end. If the total weight carried is 40 pounds, how much does each bucket weigh?

7. Two men, one stronger than the other, have to remove a block of stone weighing 270 pounds by means of a light plank whose length is 6 feet; the stronger man is able to carry 180 pounds. How must the plank be placed so as to allow him that share of the weight?

8. A horizontal rod 15 feet long can turn about one extremity which is fixed; a force of 10 dynes acts upward at the other end, and one of 20 dynes is applied downward at a point between them. Find where the 20 dyne force must be applied to maintain equilibrium.

9. The ratio of the magnitudes of two unlike parallel forces is $\frac{4}{5}$, and the distance between them is 10 inches. Find the position of the resultant.

10. The [illegible]ant of two unlike parallel forces is 2 dynes and acts at distances 6 cm. and 8 cm. from them. Find the forces.

11. A plank weighing 10 pounds rests on a single prop at its middle point; if it is replaced by two others, one on each side of it, 3 feet and 5 feet from the middle point, find the pressure on each.

12. Break up a force P into two like parallel forces in the ratio $m : n$; if one acts at a distance a from P, find the distance at which the other force acts from P.

13. A straight weightless rod 2 feet in length rests in a horizontal position between two fixed pegs placed at a distance of 3 inches apart, one of the pegs being at one end of the rod. A weight of 5 pounds is suspended at the other end. Find the pressure on each of the pegs.

14. A light rigid rod 20 feet long is supported in a horizontal position on two posts 9 feet apart, one post is 4 feet from the end of the rod; from the middle point of the rod a weight of 63 pounds is suspended. Find the pressures on the posts.

15. A uniform rod 2 feet long, whose weight is 7 pounds, is placed upon two nails, which are fixed at two points A and B in a vertical wall. AB is horizontal and 5 inches long. Assuming that the weight of the rod acts at its middle point, find the distance to which the ends of the rod extend beyond the nails, if the difference between the pressures on the nails is 5 pounds.

16. Unlike parallel forces of 3 dynes and 7 dynes act at points of a bar 10 cm. apart. Find the least length of the bar that it may be capable of being kept in equilibrium by a single force acting on it.

CHAPTER X.

MOMENTS.

1. Moment Defined.

If a rod OA is free to rotate about a fixed point O in it, and a force F act on the rod at the point A, as shown in Fig. 43, the rod will turn about O under the action of the force, unless O and A are coincident. It is evident that the power of the force to produce rotation will depend upon :—

(1) The magnitude of F, and

(2) The length of the perpendicular drawn from the point on the line of action of the force.

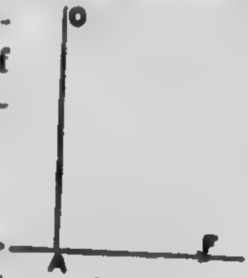


Fig. 43.

The measure of the power of the force to produce rotation about the point will, therefore, be the product of the magnitude of the force into the perpendicular drawn from the given point upon the line of action of the force.

This product is called the **Moment of the Force** with respect to the point, or,

The moment of a force about a given point is the product of the force into the length of the perpendicular drawn from the given point on the line of action of the force.

If rotation in one direction is regarded as positive, rotation in the opposite direction is negative. Rotation contra-clockwise (Fig. 43) is generally considered to be positive; but this is, of course, a mere convention.

The moment of the force about the given point vanishes only when either the force vanishes, or the line of action of the force passes through the given point.

Experiment 1.

Arrange apparatus as in the experiment on page 100, and suspend any weight, say 6 ozs., from *b*.

1. What weights must be suspended from *d* to maintain equilibrium?

2. What is the moment of each of the forces acting on the rod about each of the points *a*, *b*, *c*, *d*, *e*, *f*, and *o*?

EXERCISE XX.

1. ABCD is a square, whose side is 2 ft. long. Find the moments about both A and D of the following forces: (1) 3 pounds along AB, (2) 9 pounds along CB, (3) 2 pounds along DA, (4) 11 pounds along AC, (5) 1 pound along DB, (6) 20 pounds along DC.

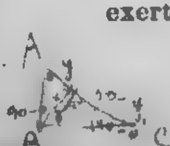
2. A force of 12 acts along a median of an equilateral triangle whose side is 18. Find the measure of the moment of the force about each angle of the triangle.

3. A force of 6 acts along one side of an equilateral triangle whose side is 10. Find the measure of its moment about the opposite angle.

4. ABCD is a rectangle, the side AB being 12 cm. and the side BC 5 cm. long. O is the intersection of the diagonals. Find the algebraic sum of the moments about (1) A, (2) O, of the following forces: 14 dynes along BA, 19 dynes along BC, 3 dynes along CD, 4 dynes along AD, 10 dynes along AC, and 9 dynes along DB.

5. A force of 20 acts along a diagonal of a square whose side is $8\sqrt{2}$. Find the measure of its moment about each of the four angles.

6. At what point of a tree must one end of a rope whose length is 50 feet be fixed, so that a man pulling at the other end may exert the greatest force to pull it over.



$\frac{x}{y} = \frac{50-y}{x}$ $x^2 = (50-y)\,y$

AB = $25\sqrt{2}$

7. ABCD is a rhombus, the side AB being 8 cm. long, and the angle ABC, 60° ; O is the intersection of the diagonals. Find the algebraic sum of the moments about (1) A, (2) O, of the following forces : 9 dynes along AB, 2 dynes along CD, 5 dynes along DA, 13 dynes along AC, 7 dynes along BC, 1 dyne along BD.

8. A and B are two points 1 metre apart ; a force of 5 dynes acts at A perpendicular to AB, and a force of 7 dynes acts at B parallel to the first force. Find the point in AB about which the moments of these forces are equal in magnitude.

9. The connecting-rod of an engine is inclined to the crank-arm at an angle of 30°. Compare the moment of the force to turn the shaft when in this position with the moment when in the most favorable position.

10 ABC is an equilateral triangle each side of which is 18 cm. long, and forces of 4 dynes and 5 dynes act at A along AB and AC respectively. Find the point in BC about which the moments of these forces are equal.

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2. Geometrical Representation of a Moment.

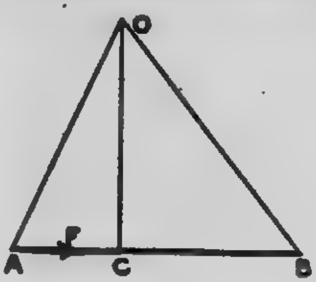


Fig. 44.

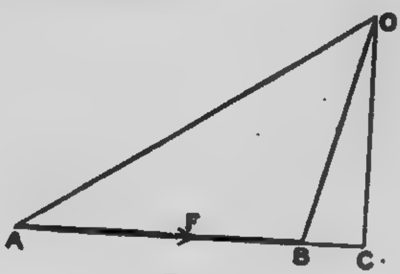


Fig. 45.

Let the given force F be represented in magnitude, direction, and line of action by AB, and let O be any given point. Draw OC perpendicular to AB (Fig. 44) or AB produced (Fig. 45). Join OA and OB.

The moment of F about O = F × OC

= AB × OC.

But AB × OC = twice the area of the triangle OAB. Therefore the moment of the force F about O is represented by twice the area of the triangle OAB. Hence, **the moment of a force about a given point is represented by twice the area of the triangle whose base is the line representing the force and whose vertex is the point about which the moment is taken.**

N.B.

3. Principle of Moments.

Experiment 2.

Arrange apparatus as in the experiment on page 100. Suspend any weight from *c*, and weights to balance it from *f* and *k*.

If P, Q and R denote the weights suspended from *c*, *f* and *k* respectively, compare the moment of R about the points *a*, *b*, *c*, *d*, *e*, *f* and *o* with the sum of the moments of P and Q about the same point. In making the calculation, include the weight of the bar in R, and take the moments in one direction as positive, and those in the opposite as negative.

Repeat the experiment several times, hanging different weights from different points.

N.B.

It will be found that in all cases **the algebraic sum of the moments of any two forces about any point in their plane is equal to the moment of their resultant about the same point.**

This proposition is generally known as the **principle of moment.** A general demonstration may be given as follows:—

(1) When the forces act at a point.

Let AP and AQ (Figs. 46 and 47) be the directions of the two forces P and Q acting at A, and AR the direction of their resultant R.

Hand In

Let O be any point in their plane.

Through O draw OD parallel to AP, meeting AQ and AR in C and D respectively.

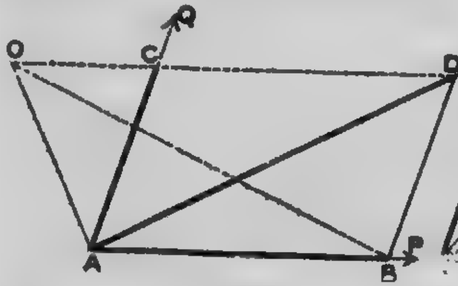


FIG. 46.

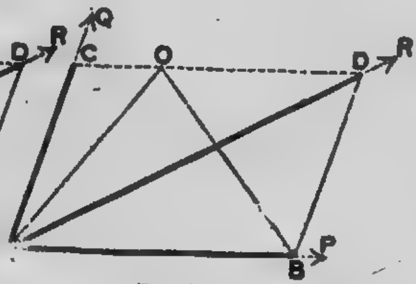


FIG. 47.

Through D draw DB parallel to AQ.

Then AB, AC, AD will completely represent P, Q, R respectively. (Art. 5, page 65).

Join OA, OB.

Then Moment of P about O $= 2\triangle OAB$

Q $= 2\triangle OAC$

R $= 2\triangle OAD$.

(1) When O is without the angle DAC, as in Fig. 46.

Moment of R $= 2\triangle OAD$

$= 2\triangle ACD + 2\triangle OAC$

$= 2\triangle DAB + 2\triangle OAC$

$= 2\triangle OAB + 2\triangle OAC$

= moment of P + moment of Q.

(2) When O is within the angle DAC, as in Fig. 47.

Moment of R $= 2\triangle OAD$

$= 2\triangle DAC - 2\triangle OAC$

$= 2\triangle DAB - 2\triangle OAC$

$= 2\triangle OAB - 2\triangle OAC$

= moment of P − moment of Q.

Hence, in either case the moment of the resultant is the algebraic sum of the moments of the forces.

(II) When the lines of action of the forces are parallel.

Let P and Q be the two parallel forces, R their resultant, and O any point in their plane about which moments are to be taken (Fig. 48).

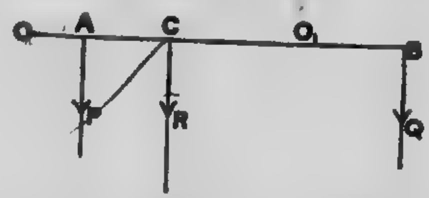


FIG. 48.

From O draw the line OACB perpendicular to the lines of action of the forces meeting them in A, B, and C, respectively.

Then $R = P + Q$

and $P.AC = Q.BC$. Art. 1, page 101.

(1) When the point O is not between the lines of action of the forces.

The moment of R about O $= R.OC$

$$= (P + Q)OC$$
$$= P.OC + Q.OC$$
$$= P(OA + AC) + Q(OB - BC)$$
$$= P.OA + Q.OB + P.AC - Q.BC$$
$$= P.OA + Q.OB, \text{ since } P.AC = Q.BC$$
$$= \text{moment of } P + \text{moment of } Q.$$

(2) When the point O is between the lines of action of the forces, or O_1.

The moment of R about $O_1 = R.O_1C$

$$= (P + Q)O_1C$$
$$= P.O_1C + Q.O_1C$$
$$= P(O_1A - AC) + Q(BC - O_1B)$$
$$= P.O_1A - Q.O_1B - P.AC + Q.BC$$
$$= P.O_1A - Q.O_1B$$
$$= \text{moment of } P - \text{moment of } Q.$$

Demonstrate the theorem for parallel forces when P and Q are unlike parallel forces.

4. Generalized Theorem of Moments.

Experiment 3.

Arrange apparatus as in the experiment on page 100. Suspend weights from the point *a* and *c*, and weights to balance them from the points *e* and *k*.

Show that the algebraic sum of the moments of the forces about any one of the points *a*, *b*, *c*, *d*, *e*, or *o* is zero.

Repeat the experiment several times, changing the number, denominations and positions of the weights.

This experiment tends to show that **if any number of forces in one plane acting on a rigid body have a resultant, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant about that point.**

The general demonstration given above may be extended to verify this proposition, as follows:—

Let P, Q, R, S, ... be the forces, and O be any point in their plane about which moments are to be taken.

Let P_1 be the resultant of P and Q,
P_2 be the resultant of P_1 and R,
P_3 be the resultant of P_2 and S,

and so on until the final resultant is obtained.

Then the moment of P_1 about O = the algebraic sum of the moments of P and Q.

Also the moment of P_2 = algebraic sum of moments of P_1 and R
= algebraic sum of moments of P, Q and R
So the moment of P_3 = algebraic sum of moments of P_2 and S
= algebraic sum of moments of P, Q, R, S.

And so on until all the forces have been taken.

If the forces are in equilibrium, their resultant is zero. Therefore the moment of the resultant about any point in their plane is zero.

Hence,

When a number of forces in one plane acting on a rigid body are in equilibrium, the algebraic sum of their moments about any point in their plane is zero.

5. Conditions of Equilibrium.

It is evident that the converse of this last proposition is true only under limitation, because there is always a series of points about any one of which the algebraic sum of the moments of any given system of forces in one plane is zero; viz., the series of points which lie in the line of action of the resultant of the given system of forces. The following is a statement of the converse.

If the algebraic sum of the moments of any number of forces in one plane about any point in their plane vanishes, then, *either*

(1) Their resultant is zero, in which case the forces are in equilibrium,

or (2) The resultant passes through the point about which the moments are taken.

The following are the **sufficient conditions of equilibrium** when a number of forces act on a rigid body in the same plane.

1. **The algebraic sum of the forces resolved in any two directions must vanish, and**

2. **The algebraic sum of the moments of the forces about any point in their plane must vanish.**

It is to be noted that **both** the above conditions are to be satisfied.

Examples.

A uniform iron rod is of length 6 feet and mass 9 pounds, and from its extremities are suspended masses of 6 and 12 pounds respectively. From what point must the rod be suspended that it may remain in a horizontal position?

Let AB (Fig. 49) represent the rod, and let the mass of

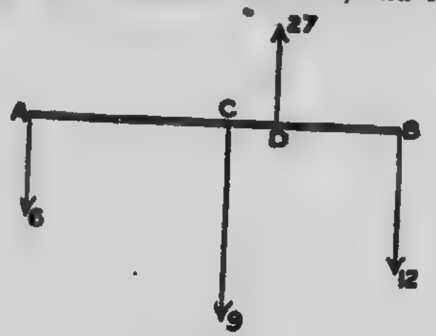


FIG. 49.

6 pounds be suspended at A and the mass of 12 pounds at B. Since the rod is uniform it may be assumed that its weight acts at C, the middle point of AB.

Let x = the distance of the point of suspension D from A.

Since the forces are parallel, the tension of the string by which the rod is suspended will be $6+9+12=27$ pounds.

The forces being in equilibrium, their resultant is zero, and the algebraic sum of moments of the forces about any point in their plane is zero.

Hence, taking moments about A,

$$-9\times 3+27x-12\times 6=0$$

$$27x=27+72=99$$

$$x=3\tfrac{2}{3}.$$

Experiment 4.

Support a uniform beam AB, whose weight is, say 9 ozs., on knife edges C and D attached to upright supports of the

form shown in Fig. 50. Suspend weights from points in the beam as shown in the figure. Determine by a calculation the re-action of the knife edges. Repeat the experiment several

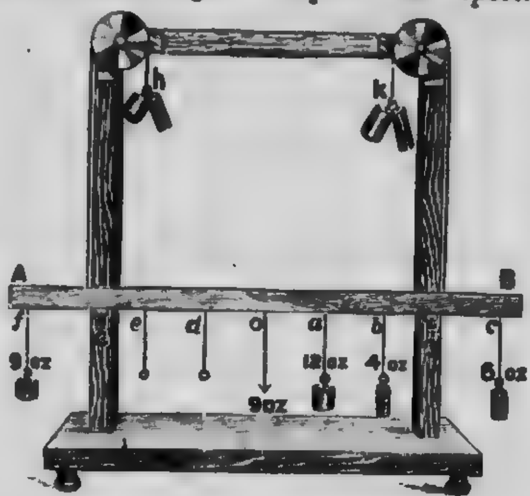


FIG. 50.

times, changing the number, denominations and positions of the weights and verifying the answers each time by suspending weights from the string running over the pulleys.

Before proceeding with the following exercise the student will find it to his advantage to work out a number of practical questions based on the use of the apparatus described in Experiments 1–4.

EXERCISE XXI.

1. A uniform beam is of length 12 metres and mass 50 kgms., and from its ends are suspended bodies of masses 20 and 30 kgms. respectively. At what point must the beam be supported that it may remain in equilibrium?

2. A lever with a fulcrum at one end is 3 feet in length. A mass of 24 pounds is suspended from the other end. If the mass of the lever is 2 pounds and acts at its middle point, at what distance from the fulcrum will an upward force of 50 pounds preserve equilibrium?

50
24 2 24

3. Three parallel forces, 10, – 15, 40, act at points 3 feet, 4 feet, 5 feet from one end of a rod and at right angles to it. Where does their resultant act?

4. Masses of 7 lbs., 1 lb., 3 lbs., and 5 lbs. are placed on a rod, supposed weightless, 1 foot apart. Find the point on which the rod will balance.

5. A bar 16 cm. long is balanced on a fulcrum at its middle. On the right arm are suspended 4 grams and 3 grams at distances of 5 cm. and 7 cm. respectively from the middle, and on the left arm 5 grams at a distance 5 cm. from the middle and w at the end. Determine w.

6. A light rigid bar 30 feet long has suspended from its middle point a mass of 700 lbs., and rests on two walls 24 feet apart, so that 1 foot of it projects over one of them. A mass of 192 lbs. is suspended from a point 2 feet from the other end. What is the pressure borne by each of the walls?

7. Six parallel forces of 7 dynes, 6 dynes, 5 dynes, 4 dynes, 3 dynes and 2 dynes are applied to a rigid rod at points 1 metre apart. Find the magnitude and position of the resultant.

8. Five parallel forces 1, 6, 3, 4, 8 dynes act 1 metre apart on a straight horizontal rod. What force must be added to the 1 dyne, in order that if the rod is supported where the 3 dynes act it may remain horizontal?

9. Four parallel forces 3, 2, 5, 7 dynes act at distances of 6 cm. apart along a straight rod and at right angles to it. Where must a force of 17 dynes act in order to maintain equilibrium?

10. A straight uniform heavy rod of length 5 feet has masses of 15 and 22 lbs. attached to its ends, and rests in equilibrium when placed across a fulcrum distant $2\frac{1}{2}$ feet from the 22lb. mass. Find the mass of the rod.

11. A straight rod 2 feet long rests in a horizontal position between two fixed pegs, placed at a distance of 3 inches apart, one of the pegs being at one end of the rod. If a mass of 5 lbs. is suspended at the other end, find the pressure on each of the pegs.

12. A uniform rod 2 feet long, whose mass is 7 lbs., is placed upon two nails, which are fixed at two points A and B in a vertical wall. AB is horizontal and 5 inches long. Find the distance to which the ends of the rod extend beyond the nails if the difference between the pressures on the nails is 5 pounds.

13. A light rod AB, 20 cm. long, rests on two pegs whose distance apart is 10 cm. How must it be placed so that the pressure on the pegs may be equal when masses of 2 W and 3 W respectively are suspended from A and B?

14. A heavy uniform beam, whose mass is 40 kgm., is suspended in a horizontal position by two vertical strings attached to the ends, each of which can sustain a tension of 35 kgm. How far from the centre of the beam must a body, of mass 20 kgm., be placed so that one of the strings may just break?

15. A heavy tapering rod, having a mass of 20 lbs. attached to its smaller end, balances about a fulcrum placed at a distance of 10 feet from the end. If the mass of the rod is 200 lbs., find the point about which it will balance when the attached mass is removed.

16. A rod 6 inches long and 1 lb. mass is supported by two vertical strings at its ends. A mass of 3 pounds is attached to the rod at a distance of 1 inch from one end. At what distance from the other end must a mass of 4 lbs. be attached in order that the tensions of the two strings may be equal?

17. A light horizontal rod 3 metres long has a mass of 15 kgm. suspended from a point on it, and it is supported by strings which apply forces to it which are in the ratio of 1:2:4:8, and which are fastened to the rod at points each 1 metre apart. Where is the mass attached, and what force does each string apply to the rod?

18. A uniform rod, whose weight is W, when suspended at a certain point rests in a horizontal position with vertical forces of W_0 and W_1 at its extremities, or W_2 and W_0 at the same ends. What vertical force at one end will keep it horizontal when suspended at the same point, W_0 being greater than W_1?

19. A rod 16 cm. long, rests on two pegs 9 cm. apart with its centre midway between them. The greatest masses that can be

suspended in succession from the two ends without disturbing equilibrium are 4 grams and 5 grams respectively. Find the weight of rod and the position of the point at which its weight [redacted]

20. A uniform bar of iron 10 feet long projects 6 feet over the edge of a wharf, there being a mass placed on the other end; and it is found that when this is diminished to 3 cwt, the bar is just on the point of falling over. Find its mass. ∴ W × 1 = 4 × 300
W = 12 cwt.

In solving problems in which it is necessary to determine the relations among the forces which are impressed in one plane on a rigid body and keep it at rest, the following rules may be found to be of value.

1. Construct a diagram of the system of forces which keep the body at rest, representing each force by a straight line and its direction by an arrow. In drawing lines to represent the lines of action of the various forces the following points may be observed.

(*a*) The re-actions of smooth surfaces are at right angles to the surfaces, for example, if a smooth beam rests against a smooth wall the re-action of the wall is at right angles to the surface.

(*b*) When three forces, not parallel, are in equilibrium, their lines of action must meet in a point. Prove.

2. Denote all unknown forces by letters.

3. Equate to zero the algebraic sum of the components of the forces in two convenient directions at right angles. These relations will furnish two equations.

In choosing the directions for resolution, the solution is generally simplified by resolving along and at right angles to the directions of unknown forces. Forces not to be determined may thus be eliminated.

4. Equate to zero the algebraic sum of the moments of the forces about some convenient point. A third equation is thus furnished. If additional equations are required, they are obtained from the geometrical relations of the figure.

In choosing the point about which moments are to be taken, it is generally advisable to choose a point common to the directions of as many forces as possible. In this way also unknown forces not to be determined may be eliminated.

In the solutions of the following examples some of the above artifices will be found to be employed.

Examples.

1. A straight rod, supposed weightless, is hinged at one end, and makes an angle of 30° with the vertical. If a mass of 7 lbs. hangs from the other end, what force acting perpendicularly to the rod at its middle point will preserve equilibrium?

Fig. 51.

Let AB (Fig. 51) be the rod hinged at A. The forces acting on it are

(1) A force of 7 lbs., acting vertically downward at B.

(2) A force P, acting at right angles to the rod at its middle point C.

(3) The re-action of the hinge, Q, acting at A. The line of action of this force will be EA, because the forces being in equilibrium, their lines of action meet in a point.

Since Q is not required, equate to zero the algebraic sum of the moments of the forces about A, a point in its line of action.

Then

$$P \times AC - 7 \times AD = 0$$

or

$$P \times \tfrac{1}{2}AB - 7 \times AB \sin 30^\circ = 0$$

$$P \times \tfrac{1}{2}AB - 7 \times \tfrac{1}{2}AB = 0$$

$$P = 7.$$

Find the re-action of the hinge.

2. A uniform beam AB, 17 feet long, whose mass is 120 lbs., rests with one end against a smooth vertical wall, and the other end on a smooth horizontal floor, this end being tied by a string 8 feet long to a peg at the bottom of the wall. Find (1) the tension of the string, (2) the re-action of the wall, (3) the re-action of the floor.

The forces acting on AB (Fig. 52) are

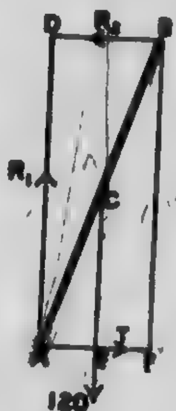


Fig. 52.

(1) Its weight, 120 lbs., acting vertically downward at its middle point C.

(2) The re-action of the floor, R_1, acting perpendicularly to the floor at A.

(3) The re-action of the wall, R_2, acting perpendicularly to the wall at B.

(4) The tension of the string, T, acting parallel to the floor at A.

Equating to zero the algebraic sum of the horizontal forces,

$$T - R_2 = 0 \quad \ldots \ldots \ldots \quad (1)$$

Equating to zero the algebraic sum of the vertical forces,

$$R_1 - 120 = 0 \quad \ldots \ldots \ldots \quad (2)$$

or

$$R_1 = 120.$$

Equating to zero the algebraic sum of the moments of the forces about A,

$$R_2 \times AD - 120 \times AE = 0 \quad \ldots \ldots \quad (3)$$

or

$$R_2 \times 15 - 120 \times 4 = 0$$

$$R_2 = 32.$$

From (1)

$$T = R_2 = 32.$$

3. A uniform ladder, 40 feet long, whose mass is 160 lbs., rests with one end on the top of a wall and is prevented from slipping by a peg driven into the ground at its lower end. If the inclination of the ladder to the horizon is 30°, find the pressure at the base and on the wall.

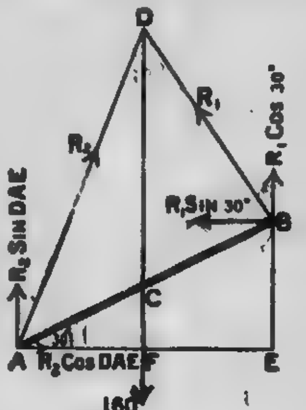


Fig. 53.

Let AB (Fig. 53) be the ladder. The forces acting on it are:—

(1) Its weight, acting vertically downward at its middle point C.

(2) The re-action of the wall, R_1, acting at right angles to the ladder (the ladder resting on the top of the wall) at B.

(3) The pressure at the peg, acting at A. The line of action of this force will be AD because the forces being in equilibrium their lines of action meet in a point.

Equating to zero the algebraic sum of the vertical components,

$$R_1 \cos 30° + R_2 \sin DAE - 160 = 0 \quad . \quad . \quad . \quad (1)$$

Equating to zero the algebraic sum of the horizontal components,

$$R_2 \cos DAE - R_1 \sin 30° = 0 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equating to zero the algebraic sum of the moments of the forces about A,

$$R_1 \times 40 - 160 \times AF = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From (3)

$$40 R_1 - 160 \times 20 \cos 30° = 0$$

or

$$R_1 = 40\sqrt{3}.$$

Transposing, and dividing (1) by (2)

$$\frac{R_2 \sin DAE}{R_2 \cos DAE} = \frac{160 - R_1 \cos 30°}{R_1 \sin 30°}$$

or

$$\tan DAE = \frac{160 - 60}{20\sqrt{3}} = \frac{5}{\sqrt{3}}.$$

Therefore, $\cos DAE = \frac{\sqrt{3}}{2\sqrt{7}}$

and, substituting in (2)

$$R_2 = 40\sqrt{7}.$$

4. A uniform rod, 16 feet long, whose mass is 100 lbs., is placed on two smooth planes whose inclinations to the horizon are 30° and 60° respectively. Find the pressure on each plane and the inclination of the rod to the horizon when in equilibrium.

Let AB (Fig. 54) be the rod. The forces acting on it are:

(1) Its weight, acting vertically downward at its middle point C.

(2) The re-action of the plane at A, acting at right angles to the plane.

(3) The re-action of the plane at B, acting at right angles to the plane.

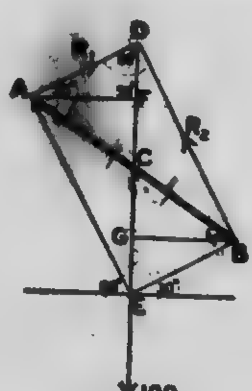


FIG. 54.

Since the forces are in equilibrium their lines of action meet in a point D.

The figure ADBE is a rectangle.

Equating to zero the algebraic sum of the moments of the forces (1) about A, (2) about B, we have

$$R_2 \times AD - 100 \times AF = 0 \quad . \quad (1)$$
$$100 \times BG - R_1 \times BD = 0 \quad . \quad (2)$$

From (1)

$$R_2 \times AD - 100 \times AD \cos 30^\circ = 0$$

or $$R^2 = 50\sqrt{3}.$$

From (2)

$$100 \times BD \cos 60^\circ - R_1 \times BD = 0$$

or $$R_1 = 50.$$

The inclination of the rod to the horizon =

$$CAF = CAD - DAF = CDA - DAF$$
$$= 60^\circ - 30^\circ = 30^\circ.$$

R

P x 2 100 √3

EXERCISE XXII.

1. A uniform rod whose mass is 60 lbs. is movable about a hinge at one end. It is kept in equilibrium in a position making an angle of 30° with the horizontal by a force making an angle of 30° with the rod at its other end. Find the re-action of the hinge and the direction of its line of action.

2. A unform rod is suspended from a peg by two strings, one attached to each end. The strings are of such lengths that the angles between them and the rod are 30° and 60° respectively. Find the tensions of the strings, the mass of the rod being one kilogram.

3. A straight lever is inclined at an angle of 60° to the horizon, and a mass of 360 lbs. hung freely at the distance of 2 inches from the fulcrum is supported by a force acting at an angle of 60° with the lever, at the distance of 2 feet on the other side of the fulcrum. Find the force.

4. A rod AB movable about a hinge A has a mass of 20 lbs. attached at B. B is tied by a string to a point C vertically above A and such that CB is six times AC. Find the tension of the string BC.

5. A heavy uniform rod AB whose mass is W is hinged at A to a fixed point, and rests in a position inclined at 60° to the horizon, being acted on by a horizontal force F applied at the lower end B. Find the re-action of the hinge and the magnitude of F.

6. A uniform rod AB of mass W free to turn about the end A which is fixed, is supported in a position inclined to the vertical by means of a string which is attached to B, and after passing over a pulley C vertically above A, supports a mass of $\frac{1}{2}$ W. If AC = BA, find the inclination of the rod to the horizontal.

7. A uniform rod AB of mass W is movable in a vertical plane about a hinge A, and is sustained in equilibrium by a mass P attached to a string BCP passing over a smooth peg C, AC being vertical. If AC = AB, show that P = W cos ACB, and that the re-action of the hinge is W sin ACB.

8. A light rod is hinged at one end and loaded at the other end with a weight of 6 pounds. The rod is supported in a horizontal position by a string which is attached to the loaded end, and which makes an angle of 30° with the rod. Find the tension of the string and the re-action of the hinge.

9. ACB is a bent lever with its fulcrum at C. The arms CA, CB are straight, equal in length, and inclined to each other at an angle of 135°. When CA is horizontal a mass of P attached at A sustains a mass of W attached at B; and when CB is horizontal the mass of W at B requires a mass Q attached at A to balance it. Find the ratio of P to Q.

10. ACB is a bent lever with its fulcrum at C. The angle ACB is a right angle, the arm AC is 10 feet and BC 7 feet long, and AC is in a vertical position. If a horizontal force of 21 pounds acting at A is balanced by a vertical force P, acting at B, find the magnitude of P and the pressure on the fulcrum.

11. A uniform beam, 32 feet long, whose mass is 200 lbs., rests with one end on a smooth horizontal plane and the other end against a smooth vertical wall. If a string, 16 feet long, connects the lower end with the foot of the wall, find (1) the tension of the string, (2) the pressure against the wall, (3) the pressure on the plane.

12. A ladder, the weight of which is 90 pounds, acting at a point one-third of its length from the foot, is made to rest against a smooth vertical wall, and inclined to it at an angle of 30°, by a force applied horizontally at the foot. Find the force.

13. A uniform ladder, 40 feet long, whose mass is 180 lbs., rests with one end against a smooth vertical wall and is prevented from slipping by a peg in the ground. Find the pressures against the wall and at the ground if the inclination of the ladder to the horizon is 60°.

14. ACB is a uniform rod, of mass W; it is supported (B being uppermost) with its end A against a smooth vertical wall AD by means of a string CD, DB being horizontal and CD inclined to the wall at an angle of 30°. Find the tension of the string and the pressure on the wall, and prove that AC = $\frac{1}{3}$ AB.

15. A uniform beam, 12 feet long, whose mass is 50 lbs., rests with one end A at the bottom of a vertical wall, and a point C in the beam 10 feet from A is connected by a horizontal string CD with a point D in the wall 8 feet above A. Find (1) the tension of the string, (2) the pressure against the wall.

16. A ladder, 14 feet long, whose mass is 50 lbs., rests with one end against the foot of a vertical wall; and from a point 4 feet from the upper end a cord which is horizontal runs to a point 6 feet above the foot of the wall. Find the tension of the cord and the re-action at the lower end of the ladder.

17. A uniform heavy beam AB, whose mass is W, rests against a smooth horizontal plane CA and a smooth vertical wall CB, the lower extremity A being attached to a string which passes over a smooth pulley at C and sustains a mass P. Find the pressure on the plane and the wall.

18. A uniform rod AB, whose mass is 100 lbs., is inclined at an angle of 60° to the vertical with one end A resting against a smooth vertical wall, being supported by a string attached to a point C of the rod, distant 1 foot from B, and also to a ring in the wall vertically above A. If the length of the rod is 4 feet, find the position of the ring and the inclination and tension of the string.

19. A uniform ladder rests against a smooth wall, the ground being also smooth. Compare the horizontal forces which must be applied to the bottom of the ladder to preserve equilibrium, when a weight equal to the weight of the ladder is placed on the ladder at the top and bottom respectively.

20. A uniform ladder, 36 feet long, rests with one end on a smooth wall, and the lower end is prevented from slipping by a peg. If the inclination of the ladder to the horizon is 30°, find the pressure on the wall and on the peg, the mass of the ladder being 100 lbs.

21. A uniform ladder, whose mass is 20 kgm., rests with its lower end upon a smooth horizontal plane, and its upper end on a slope inclined at an angle of 60° to the horizon; the ladder makes an angle of 30° with the horizon. Find the pressure on the plane and the slope respectively and the force which must act horizontally at the foot of the ladder to prevent sliding.

22. A ladder, the weight of which may be regarded as a force acting at a point one-third of the length from the foot, rests with one end against a peg in a smooth horizontal plane, and the other end on a wall. The point of contact with the wall divides the ladder into parts which are as 1:4. If the mass of the ladder is 120 lbs., and it makes an angle of 45° with the horizontal plane, find the pressure on the peg and the re-action of the wall.

23. The lower end of a uniform pole rests on the ground, and a point 2 feet from its upper end rests against a smooth rail, the pole being inclined at an angle of 60° to the horizon. If the length of the rod is 7 feet and its mass 21 lbs., find the direction and magnitude of the re-action of the ground on the pole.

24. A carriage wheel, whose mass is W and radius r, rests upon a level road. Show that the least force F which will be on the point of drawing the wheel over an obstacle of height h is

$$F = \frac{W\sqrt{(2rh - h^2)}}{r - h}$$

25. A spherical shot, whose mass is 60 lbs., rests between two planes which are inclined at angles of 30° and 60° to the horizon. Find the pressure on each plane.

26. A spherical shot, whose mass is 30 kgm., rests between a smooth vertical wall and a smooth plane, the inclination of the latter to the horizon being 45°. Find the pressure on the wall and the plane.

27. A smooth sphere of radius a and mass W is supported on a smooth plane inclined at an angle of 30° to the horizon by a string, one end of which is fastened to a point on the plane and the other end to the surface of the sphere. If in the position of equilibrium the string is horizontal, find the length of the string and the pressure on the plane.

28. A solid sphere rests on two parallel bars which are in the same horizontal plane, the distance between the bars being equal to the radius of the sphere. If the mass of the sphere is 90 lbs., find the re-action of each bar.

29. Two smooth spheres, the mass of each of which is 10 kgm., are strung on a thread which is then suspended by its extremities so that the upper portions are parallel. Find the pressure between the spheres, the holes being smooth.

30. Two spheres, each of mass W and radius r, rest inside a hollow sphere, of radius $3r$. Find the pressure between (1) the two spheres, (2) a solid sphere and the hollow one.

31. A smooth sphere, whose mass is 9 kgm., is supported in contact with a smooth vertical wall by a string fastened to a point on its surface, the other end being attached to a point in the wall. If the length of the string is equal to the radius of the sphere, find (1) the inclination of the string to the vertical, (2) the tension of the string, (3) the re-action of the wall.

32. A ring, mass 9 lbs., slides freely on a string of length $a\sqrt{2}$ whose ends are fastened to two points at a distance a apart in a line making an angle of 45° with the horizon. Find the tension of string in the position of equilibrium.

33. Two posts, one of which is $a(\sqrt{3}-1)$ feet higher than the other, stand at a horizontal distance $a(\sqrt{3}+1)$ feet apart. A body whose mass is 18 lbs. hangs by two strings, of length $2a\sqrt{2}$ feet, attached each to the top of one of the posts. Find the tensions of the strings.

34. A string is tied to two points. A ring, mass W, can slip freely along the string, and is pulled by a horizontal force P. If the parts of the string when in equilibrium are inclined at 30° and 45° respectively to the horizon, find the value of P.

35. If a string ACDB is 21 inches long; C and D two points in it such that AC=6 inches, CD=7 inches; and if the extremities be fastened to two points in the same horizontal line at a distance of 14 inches from each other; what must be the ratio of the two masses, which, hung at C and D, will keep CD horizontal?

36. A horizontal rod is supported by two strings, each 1 yard long, passing over a smooth peg placed 1 foot vertically above the middle of the rod. The ends of each string are attached respectively to one end and the middle of the rod. Show that the tension of each string is one-third the weight of the rod.

37. A uniform rod, 10 feet long, is placed on two smooth planes whose inclinations to the horizon are 30° and 60° respectively. Find the pressure on each plane and the inclination of the rod to the horizon when in equilibrium, the mass of the rod being 40 lbs.

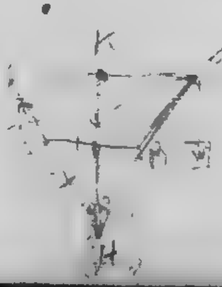
38. A uniform rod AB, 18 feet long and of 20 lbs. mass, is hinged at A, and a mass of 5 lbs. is suspended from B. It is kept at rest by a string 13 feet long, one end of which is attached to a point D on the rod 13 feet from A, and the other end to a point O 10 feet vertically above A. Find the tension of the string and the re-action of the hinge.

39. Two equal rods OA, OB, each $16\frac{1}{4}$ feet long and mass 10 lbs., are connected at O, and their ends are placed on a smooth horizontal plane, A, B, O being in the same vertical plane. If a string $20\frac{5}{8}$ feet long connect A and B, find (1) the pressure at O, (2) the pressure at A and B, (3) the tension of the string.

40. Two legs of a light step ladder are connected by a smooth joint at the top and a cord at the bottom. The ladder stands on a smooth floor with one leg, which is 3 feet long, vertical. A man, whose mass is 180 lbs., stands on the other leg at a height of 2 feet above the ground. Find the pressure on the vertical leg and the tension on the cord.

41. To upper end A of a heavy uniform rod CA, which can turn freely about a hinge C, is attached a string which passes over a smooth pulley P (the distance CP being horizontal and equal to CA), and supports a heavy particle whose mass is half that of the rod. Show that the rod can rest at an angle of 30° to the vertical, and determine the re-action of the hinge if the mass of the rod is 100 lbs.

42. A uniform beam of mass 3 tons is suspended in a horizontal position by two ropes attached at the ends; one of the ropes, of the same length as the beam, is attached to a peg; the other rope passes over a pulley and is attached to a mass W; the pulley is fixed in the same horizontal line as the peg, and at a distance from it equal to twice the length of the beam. Find W.



CHAPTER XI.

CENTRE OF GRAVITY.

1. Centre of Parallel Forces.

It was shown (Art. 1, page 101) that if two parallel forces act at points A and B the line of action of their resultant divides AB at C inversely as the forces. It is evident that so long as the forces remain parallel the position of C will not be altered by deflecting at the points of application of the forces their lines of action through any angle.

In general, the point of application of the resultant of any number of parallel forces having fixed points of application will not be altered if the lines of action of the forces be deflected through any angle at their points of application, provided that the lines of action remain parallel.

This point is called **the centre of the parallel forces.**

The centre of any number of parallel forces having fixed points of application is the point through which the direction of their resultant passes, whatever be the directions of the parallel forces.

2. Centre of Mass.

If we conceive of a system of parallel forces impressed respectively on each of the particles of a rigid body, each force being proportional to the mass of the particle on which it is impressed, the centre of the parallel forces is called **the centre of mass of the body.**

It is evident (Art. 1 above) that the position of the centre of mass depends only on the amounts of the parallel forces and the points at which they are impressed, and is independent of their direction; that is, so long as the forces remain parallel and proportional to the masses of the particles on which they are impressed, the line of action of their resultant passes through the same point in the body, whatever be its position.

3. Centre of Gravity.

There is a mutual attraction between every particle of matter and the earth, and the amount of this attraction, the weight of the particle, is proportional to its mass (Art. 6, page 28).

Now any body may be regarded as an agglomeration of particles. The weight of the body is the resultant of the weights of its particles.

If the body is small compared with the earth, the lines joining its constituent particles with the centre of the earth are approximately parallel. The weights of the particles, therefore, form a system of like parallel forces; and, as these forces are proportional to the masses of the particles, their centre will be the centre of mass of the body (Art. 2, above).

This point is called **the centre of gravity of the body.**

The centre of gravity of a body, or system of particles rigidly connected together, is that point through which the line of action of the weight of the body always passes in whatever position the body is placed.

4. To find the Centre of Gravity of a Uniform Straight Rod.

Let AB (Fig. 55) be the uniform rod. The centre of gravity is evidently the middle point, G, of the rod,

FIG. 55.

because the rod may be regarded as made up of equal particles equidistant from this point and the centre of gravity of each pair of particles, for example of M and N, is at the middle point of the line joining them, that is, at G.

5. To find the Centre of Gravity of a Uniform Parallelogram.

Let ABCD (Fig. 56) be the parallelogram, composed of some material of uniform thickness and density. Let

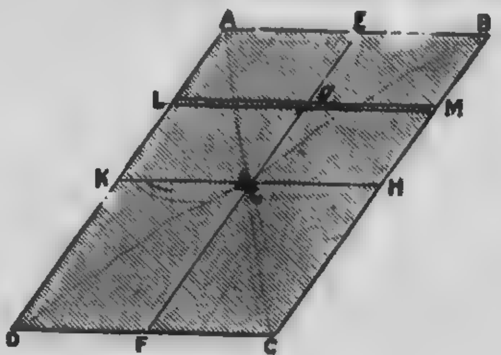


FIG. 56.

the middle points of AB, BC, CD, and DA be E, H, F, and K respectively.

Consider the lamina to be made up of a series of very thin parallel rods, such as LM, each parallel to AB and DC.

The centre of gravity of any one of these rods, LM, is at its middle point g; but EF bisects all such rods;

therefore the centre of gravity of each rod lies in EF hence the centre of gravity of the parallelogram lies in EF.

In a similar manner the parallelogram may be regarded as made up of a series of rods parallel to AD and BC, and its centre of gravity shown to lie in the line HK joining their centres.

Hence the centre of gravity of the parallelogram is at G, the point of intersection of EF and HK, the diameters of the parallelogram.

6. To find the Centre of Gravity of a Triangular Lamina.

Let ABC (Fig. 57) be the triangular lamina, and let the middle points of AB, BC and CA be D, E and F respectively.

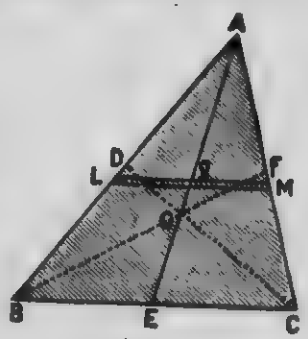


FIG. 57.

Consider the lamina to be made up of a series of very thin parallel rods such as LM, each parallel to BC.

The centre of gravity of each of these rods is at its middle point g; but the median AE bisects all such rods; therefore the centre of gravity of each rod lies in AE; hence the centre of gravity of the lamina lies in AE.

In a similar manner the lamina may be regarded as made up of a series of rods parallel to AB or AC, and its centre of gravity shown to lie in the medians CD or BF.

Hence the centre of gravity of the triangular lamina is at G, the point where the medians intersect, that is in the line joining the middle point of any side to the opposite vertex at one-third its length from that side.

EXERCISE XXIII.

1. An isosceles triangle has its equal sides of length 5 cm. and its base of length 6 cm. Find the distance of the centre of gravity from each of the angular points.

2. If the angular points of one triangle lie at the middle points of the sides of another, show that the centres of gravity of the two are coincident.

3. The equal sides of an isosceles triangle are 10 feet, and the base is 16 feet in length. Find the distance of its centre of gravity from each of the sides.

4. The sides of a triangle are 3, 4, and 5 feet in length. Find the distance of the centre of gravity from each side.

5. The sides of a triangular lamina are 6, 8, and 10 feet in length. Find the distance of the centre of gravity from each of its angular points.

6. The sides AB, AC of a triangle ABC, right-angled at A, are respectively 18 and 12 inches long. Find the distance of the centre of gravity from C.

7. Show that the centre of gravity of a lamina in the form of a parallelogram is at the point of intersection of its diagonals.

8. D is the middle point of the base BC of a triangle ABC. Show that the distance between the centres of gravity of the triangles ABD and ACD is $\frac{1}{3}$ BC.

$G_1G_2 = \frac{2}{3}EF$

$= \frac{1}{3}BC$

9. If a parallelogram is divided into four triangles by its diagonals, and the centres of gravity of these triangles are joined, show that these joining lines form a parallelogram.

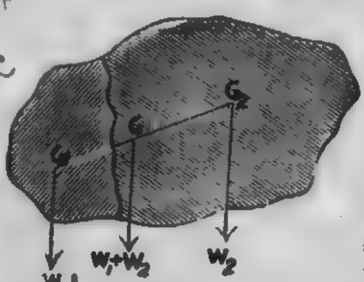


FIG. 58.

10. If the centre of gravity of a triangle coincides with the centre of gravity of the inscribed circle, show that the triangle is equilateral.

11. Show that the locus of the centres of gravity of all right-angled triangles which can be described on the same hypotenuse is a circle whose radius is one-sixth the hypotenuse.

7. Given the weight and the centres of gravity of the two portions into which a body is divided, to find the centre of gravity of the whole.

Let W_1, W_2 be the weights of the portions, and G_1, G_2 their centres of gravity. (Fig. 58.)

Then the centre of gravity of the whole is the centre of the two parallel forces W_1, W_2, and is therefore at a point G in the line $G_1 G_2$ such that

$$G_1G : GG_2 :: W_2 : W_1 \quad \text{(Art. 1, page 101)}$$

or

$$G_1G = \frac{W_2}{W_1} \times GG_2.$$

In solving the problems in the following exercises, it is to be noted that the weights of uniform laminæ are proportional to their areas.

EXERCISE XXIV.

1. An equilateral triangle is described upon one side of a square whose side is 16 inches. Find the distance of the centre of gravity of the figure so formed from the vertex of the triangle, the vertex being without the square.

2. The length of one side of a rectangle is double that of an adjacent side, and on one of the longer sides an equilateral triangle is described externally. Find the centre of gravity of the whole.

3. A piece of cardboard is in the shape of a square ABCD with an isosceles right-angled triangle described on the side BC. If the side of the square is 12 inches, find the distance of the centre of gravity of the cardboard from the line AD.

4. An isosceles right-angled triangle is described externally on the side of a square as hypotenuse. Find the centre of gravity of the whole figure.

5. A square is described on the base of an isosceles triangle. What is the ratio of the altitude of the triangle to its base when the centre of gravity of the whole figure is at the middle point of the base?

6. Two isosceles triangles are on the same base but on opposite sides of it, and the altitude of the one is 6 inches and of the other 2 inches. Find the distance of the centre of gravity of the whole figure from the common base.

7. Two triangles are on the same base and between the same parallels, prove that the distance between their centres of gravity is one-third the distance between their vertices.

8. A cross is made up of six equal squares. Find its centre of gravity.

9. A uniform rod, 1 foot in length, is broken into two parts of lengths 5 and 7 inches which are placed so as to form the letter T, the longer portion being vertical. Find the centre of gravity of the system.

10. Two rectangular pieces of cardboard, of lengths 6 and 8 inches and breadths 2 and $2\frac{1}{2}$ inches respectively, are placed, touching but not overlapping each other, on a table to form a T-shaped figure, the former piece forming the cross-bar. Find the centre of gravity.

11. ABCD is a square whose side $= 2a$. On CD, as base, an isosceles triangle CED is described externally, whose altitude $= b$. Find the distance of the centre of gravity of the whole figure from AB.

12. Two squares, of which one is four times the other, are placed so that the sides about an angular point of the one are co-linear with those about an angular point of the other. Find the centre of gravity of the figure so formed.

13. Squares are described on the three sides of an isosceles right-angled triangle, outside the triangle. Find the centre of gravity of the figure so formed.

14. Prove that the centre of gravity of the two complements which are about the diagonal of any parallelogram is in that diagonal.

15. Find the centre of gravity of a quadrilateral, two of whose sides are parallel to each other, and respectively 6 inches and 14 inches long, while the other sides are 8 inches long.

16. The sides of a quadrilateral lamina are 3, 5, 4 and 10 respectively. The side 5 is parallel to 10. Find the distance of the

Find ⊥ distance on sides or sides produced

centre of gravity of the quadrilateral from each of the sides 3 and 4.

17. ABCD is a trapezium, the angles at B and C being right angles. Show that the distance of the centre of gravity from BC is

$$\frac{AB^2+AB.CD+CD^2}{3(AB+CD)}.$$

18. If a and b denote the parallel sides of a trapezium, show that the centre of gravity of the figure lies on the line joining the points of bisection of a and b, and divides it in the ratio $2a+b:2b+a$.

19. ABCD is a trapezium, AB is parallel to CD, and $AB=\frac{1}{2}CD$. Show that the distance of the centre of gravity of the figure from AB is $1\frac{1}{4}$ times that from CD.

20. In a quadrilateral ABCD, the sides AB, AD are 15 inches, and BC, CD are 20 inches. If BD = 24 inches, find the distance of the centre of gravity from A. ✓

21. The sides of a five-sided board ABCDE are each $=a$, and the angles A and E are right angles. Prove that the distance of the centre of gravity from the side $AE=\frac{14+3\sqrt{3}}{26}a$. ✓

22. G is the centre of gravity of a triangle ABC. A line is drawn from G parallel to BC cutting AB, AC in P and Q. Show that the centre of gravity of PBCQ divides GD in the ratio of 8:7, D being the middle point of BC.

23. ABCD is a square plate, E and F being the middle points of the sides AB and BC; the plate is bent along EF so that the triangle EBF lies flat on the other side of the plate. Find the centre of gravity. c ✓

8. Given the weight and centre of gravity of a body, and the weight and centre of gravity of a portion of it, to find the centre of gravity of the remaining portion.

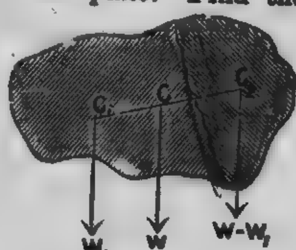


FIG. 59.

Let W be the weight of the body and G its centre of gravity, and let W_1 be the weight and G_1 the centre of gravity of the given portion. (Fig. 59.)

Then if G_2 be the centre of gravity of the remainder it must be in G_1G produced at such a distance that

$$(W - W_1) \times GG_2 = W_1 \times GG_1$$

or

$$GG_2 = \frac{W^1}{W - W_1} \times GG_1.$$

EXERCISE XXV.

1. ABCD is a square whose middle point is E and whose side $= a$. If the triangle ECD is removed, find the centre of gravity of the remainder.

2. E and F are the middle points of the sides AB, AC of an equilateral triangle ABC. If the portion AEF is removed, find the centre of gravity of the remainder.

3. ABCD is a square, O its centre, E and F the middle points of AB, AD. If AEF is cut away, find G, the centre of gravity of the remainder.

4. From a square piece of paper ABCD a portion is cut away in the form of an isosceles triangle whose base is AB and altitude equal to one-third AB. Find the centre of gravity of the remaining portion.

5. ABCD is a rectangle, E the middle point of CD; the triangle ADE is cut away. Find the centre of gravity of the remainder.

6. From a rectangular lamina the triangle formed by joining its centre of gravity G to the ends of one of the sides is cut away. Find the distance of the centre of gravity of the remaining part from the point G, when a is the length of the adjacent side.

7. An equilateral triangle has each side $= 4$ inches. From the corner A an equilateral triangle is cut off, having a side $= 1$ inch. Find the distance from A of the centre of gravity of the remainder.

8. G is the centre of gravity of a plane lamina in the form of an isosceles triangle right-angled at A, and having the side BC of length a. If the portion GBC is cut away, find the distance of the centre of gravity of the remaining piece from A.

9. If three equal triangles are cut off from a given triangle by lines drawn parallel to the sides, prove that the centre of gravity of the remaining hexagon will coincide with that of the original triangle.

10. ABC is a triangle, D is a fixed point in BC. If a triangle PBC is cut away whose vertex P is in AD, prove that whatever be the position of P the centre of gravity of the remainder lies on a fixed line.

11. A quarter of a triangle is cut off by a line drawn parallel to one of its sides bisecting each of the other sides. Find the centre of gravity of the remainder.

12. The vertex of a triangle is cut off by a line drawn parallel to the base, and the height of the figure is thus diminished by one-third. Find the centre of gravity of the remainder.

13. From the corner of a square piece of cardboard whose side is 6 inches another square whose side is 2 inches is cut away. Find the centre of gravity of the remaining piece.

14. Through the centre of gravity of a triangle ABC, a line DE is drawn parallel to the base BC. Find the centre of gravity of the figure DBCE.

15. A circular hole 1 foot in radius is cut out of a circular disc 3 feet in radius. If the centre of the hole is 18 inches from that of the disc, find the centre of gravity of the remainder.

16. Out of a circle of radius 12 inches is cut another circle whose diameter coincides with the radius of the first. Find the centre of gravity of the remainder.

17. A circular board of radius a has a hole of radius b cut out of it. Show that the centre of gravity of the remainder must lie within a circle whose radius is $\frac{b^2}{a+b}$.

18. Where must a hole, of 1 foot radius, be punched out of a circular disc, of 3 feet radius, so that the centre of gravity of the remainder may be 2 inches from the centre of the disc?

19. A circular board has two circular holes cut in it, the centres of these holes being in the middle points of two radii of the board

at right angles to each other. If the radius of each hole is one-third the radius of the board, find the centre of gravity of the remainder.

20. A uniform plate of metal, 10 inches square, has a hole of area 3 square inches cut out of it, the centre of the hole being in a diameter of the plate at a distance of $2\frac{1}{2}$ inches from the centre. Find the centre of gravity of the remainder.

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9. To find the centre of gravity of a number of particles in a straight line.

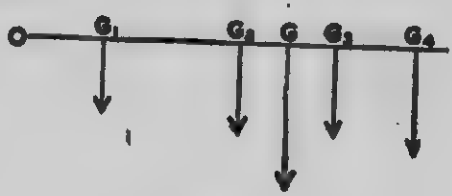


FIG. 60.

Let G_1, G_2, . . . G_n (Fig. 60) be the positions of the particles and m_1, m_2, . . . m_n their masses. Take any point O in the line, and let x_1, x_2 . . . x_n be the distances of the particles, and $\bar{x}$ the distance of their centre of gravity, G, from O.

Then G is the point of application of the resultant of a series of parallel forces proportional to m_1, m_2, . . . m_n, acting at G_1, G_2, . . . G_n.

The moment of the resultant of these forces about O is equal to the algebraic sum of the moments of the forces about the same point. (Art. 4, page 115.)

Hence,

$$(m_1+m_2+\ .\ .\ .\ +m_n)\bar{x}=m_1x_1+m_2x_2+\qquad +m_nx_n.$$

or

$$\bar{x}=\frac{m_1x_1+m_2x_2+\ .\ .\ .\ m_nx_n}{m_1+m_2+\ .\ .\ .\ m_n}.$$

EXERCISE XXVI.

1. Masses of 2 lbs., 4 lbs., 6 lbs., 8 lbs., are placed so that their centres of gravity are in a straight line, and six inches apart. Find the distance of their common centre of gravity from that of the largest mass.

2. Two masses of 6 lbs. and 12 lbs. are suspended at the ends of a uniform horizontal rod, whose mass is 18 lbs. and length 2 ft. Find the centre of gravity.

3. A rod, 1 foot in length and mass 1 ounce, has an ounce of lead fastened to it at one end, and another ounce fastened to it at a distance from the other end equal to one-third of its length. Find the centre of gravity of the system.

4. Four masses of 3 lbs., 2 lbs., 4 lbs., and 7 lbs., respectively, are at equal intervals of 8 inches on a lever without weight, 2 feet in length. Find where the fulcrum must be, in order that they balance.

5. A uniform bar, 3 feet in length and of mass 6 ounces, has three rings, each of mass 3 ounces, at distances 3, 15, and 21 inches from one end. About what point of the bar will the system balance?

6. A ladder, 50 feet long and mass 100 lbs., is carried by two men; one lifts it at one end, and the other at a point 2 feet from the other end. The first carries two-thirds of the weight which the second does. Where is the centre of gravity of the ladder?

7. A pole, 10 feet long and mass 20 lbs., has a mass of 12 lbs. fastened to one end. The centre of gravity of the whole is 4 feet from that end. Where is the centre of gravity of the pole?

8. Four masses, 1 lb., 4 lbs., 5 lbs. and 3 lbs., respectively, are placed 2 feet apart on a rod 6 feet long, whose mass is 3 lbs. and centre of gravity 2 feet from the end at which the 1 lb. is placed. Find the centre of gravity of the whole.

9. A cylindrical vessel whose mass is 4 lbs. and depth 6 inches will just hold 2 lbs. of water. If the centre of gravity of the vessel when empty is 3.39 in. from the top, determine the position of the centre of gravity of the vessel and its contents when full of water.

10. A cylindrical vessel, without lid, one foot in diameter and one foot in height, is made of thin sheet metal of uniform thickness. If it is half filled with water, where will be the common centre of gravity of the vessel and the water, assuming the mass of the vessel to be one-fifth the mass of the contained water?

---

10. To find the centre of gravity of a number of particles in a plane.

Let G_1, G_2, . . . G_n (Fig. 61) be the positions of the particles, and m_1, m_2, . . . m_n, their masses.

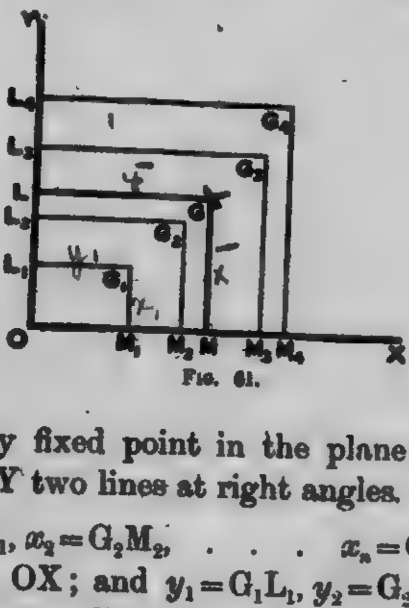


Fig. 61.

Let O be any fixed point in the plane of the forces, and OX and OY two lines at right angles.

Let $x_1 = G_1M_1$, $x_2 = G_2M_2$, . . . $x_n = G_nM_n$, the perpendiculars on OX; and $y_1 = G_1L_1$, $y_2 = G_2L_2$, ... $y_n = G_nL_n$, the perpendiculars on OY.

Let G be the centre of gravity of the particles, and let $\bar{x} = GM$, the perpendicular on OX; and $\bar{y} = GL$, the perpendicular on OY.

Then G is the point of application of the resultant of a series of parallel forces proportional to m_1, m_2, m_n, acting at G_1, G_2, . . . G_n, respectively.

The point of application of the resultant is the same whatever be the direction of the forces, provided that they remain parallel. Let them act perpendicular to the plane of the paper.

The moment of the resultant about OX and OY is equal to the algebraic sum of the moments of the forces about these lines.

Hence, taking moments about OX

$$(m_1+m_2+\ \dots\ m_n)\bar{x}=m_1x_1+m_2x_2+\qquad+m_nx_n$$

or

$$\bar{x}=\frac{m_1x_1+m_2x_2+\ \dots\ +m_nx_n}{m_1+m_2+\ \dots\ m_n}$$

and, taking moments about OY.

$$(m_1+m_2+\qquad+m_n)\bar{y}=m_1y_1+m_2y_2+\qquad+m_ny_n$$

or

$$\bar{y}=\frac{m_1y_1+m_2y_2+\ \dots\ +m_ny_n}{m_1+m_2+\ \dots\ +m_n}.$$

The position of G is determined from these two equations.

Example.

Four heavy particles whose masses are 4, 6, 5 and 3 lbs. respectively, are placed at the corners of a square plate whose sides are 26 inches, and mass 8 lbs. Find the distance of the centre of gravity of the whole from the centre of the plate.

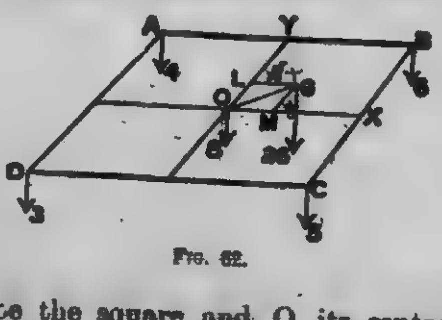


FIG. 62.

Let ABCD be the square and O its centre, and let the masses be placed as shown in Fig. 62.

Let G be the centre of gravity, and let x be its distance from OY, a line drawn through O parallel to AD and BC; and y its distance from OX, a line drawn through O parallel to AB and DC.

Then, taking moments about OY,

$$(4+6+5+3+8)\bar{x} = 8\times 0+6\times 13+5\times 13-4\times 13-3\times 13$$

or

$$x = 2$$

Again, taking moments about OX,

$$(4+6+5+3+8)y = 8\times 0+4\times 13+6\times 13-3\times 13-5\times 13$$

$$y = 1$$

Hence LG = 2, and GM = 1, and OG, the distance of the centre of gravity of the whole from the centre of the square, $=\sqrt{5}$ inches.

EXERCISE XXVII.

1. Masses of 1, 1, 1 and 2 lbs. are placed at the angular points of a square. Find their centre of gravity.

2. Masses of 2 lbs., 1 lb., 2 lbs., 3 lbs., are placed at A, B, C, D respectively, the angular points of a square. Find the distance of the centre of gravity from the centre O.

3. Masses of 1, 4, 2, 3 lbs. are placed at the corners A, B, C, D of a rectangle; a mass of 10 lbs. is also placed at the intersection of the diagonals. If AB = 7 inches, and BC = 4 inches, find the distance of the centre of gravity of the whole from A.

4. At the angular points of a square, taken in order, there act parallel forces in the ratio 1 : 3 : 5 : 7. Find the distance from the centre of the square of the point at which their resultant acts.

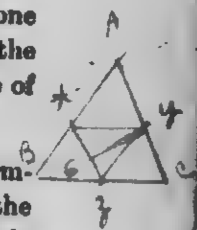
5. Masses 5, 7, 10 are placed at the three angles of a square whose side = 4 ft. Find the distance of their centre of gravity from 5.

6. Three masses 3, 4, 5 lbs. are placed at the angles of an equilateral triangle whose sides are 12 inches. Find the distance of the centre of gravity of the whole from the least mass.

7. ABC is a triangle right-angled at A, AB being 12 and AC 15 inches in length. Masses in the ratio 2 : 3 : 4 are placed at A, C, and B respectively. Find the distances of their centre of gravity from B and C.

8. Prove that the centre of gravity of an equilateral triangular lamina coincides with that of three equal masses placed at its angular points.

9. Prove that the centre of gravity of three particles, placed one at each of the angular points A, B, C of a triangle such that the mass of each is proportional to the opposite side, is at the centre of the circle inscribed in the triangle.



10. Show that the centre of gravity of three uniform rods forming a triangle ABC is at the centre of the inscribed circle of the triangle formed by joining the middle of the sides of the triangle ABC.

$\frac{2}{x} = \frac{xG}{OZ} = \frac{BC}{AB} = \frac{XY}{ZY}$

11. Three particles placed at the angular points A, B, C of a triangle are proportional to the areas of the triangles OBC, OCA, OAB respectively, where O is the centre of the circumscribing circle. Show that their centre of gravity is O.

12. If masses be placed at the angular points of a triangle respectively proportional to the sum of the sides which meet at those points, prove that their centre of gravity will coincide with that of the perimeter of the triangle.

13. ABC is a uniform triangular plate of mass w. Masses of $5w$, w and w are placed at A, B, and C respectively. If G is the centre of gravity of the triangle, show that the whole will balance about a point O such that AO = $\frac{1}{2}$ AG.

14. ABC is an equilateral triangle of side 2 feet. At A, B and C are placed masses proportional to 5, 1, 3, and at the middle points of the sides BC, CA, AB, masses proportional to 2, 4, 6. Show that their centre of gravity is distant 16 inches from B.

15. Show that if the centre of gravity of a plane quadrilateral coincides with that of four equal particles placed at the angular points the quadrilateral is a parallelogram.

16. A regular hexagon is inscribed in a circle, and masses of 1 lb. each are placed at five of the angular points of the hexagon, and 3 lbs. at the centre of the circle. Find the centre of gravity of the system.

17. A uniform bar 8 feet long is bent so as to form four of the sides of a regular hexagon. Find the distance of the centre of gravity from the centre of the circumscribing circle.

---

11. If a body is suspended freely from one point the centre of gravity of the body is in the vertical line passing through the point of suspension.

The forces acting on the body are

FIG. 63.

FIG. 64.

(1) The weight of the body acting vertically through the centre of gravity G (Figs. 63, 64).

(2) The force exerted at the point of support O.

Since these two forces are in equilibrium their lines of action must coincide. Hence the point of support must be in the same vertical line as the centre of gravity.

12. Stable, Unstable and Neutral Equilibrium.

It is evident that if the centre of gravity of the body is vertically below the point of support (Fig. 63) and the body is slightly displaced, it will tend to return to its original position. In this case the equilibrium is said to be **stable**.

If the centre of gravity of the body is vertically above the point of support (Fig. 64) the body, if displaced, will not return to its original position. The equilibrium is then said to be **unstable**.

In both of the above cases the forces acting on the body in its new position are not in equilibrium but have a resultant which, in the first case, tends to restore the body to its original position, and in the second to move it farther from that position.

If the forces acting on the body in its displaced position are in equilibrium, the body tends neither to return nor to recede. The equilibrium is then said to be **neutral**.

For example, a cone standing with its circular base on a horizontal plane is in stable equilibrium; if balanced with its vertex on the plane, it is in unstable equilibrium; while if placed with its slant side in contact with the plane, it is in neutral equilibrium.

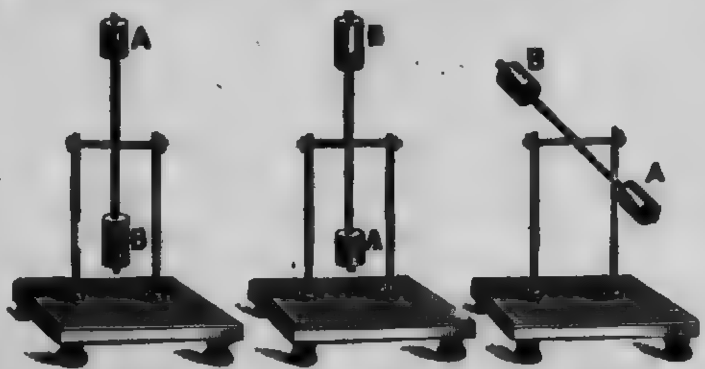


FIG. 65a. FIG. 65b. FIG. 65c.

The conditions of stable, unstable and neutral equilibrium can be well illustrated by the use of the apparatus shown in Fig. 65. When the weight A and B

are so adjusted that the centre of gravity of the combination is below the axis as in Fig. 65*a*, and the weights are displaced, they tend to return to their original positions. When they are so adjusted that the centre of gravity is above the axis, as in Fig. 65*b*, the slightest displacement of the weights causes them to move away from their positions of rest. If the weights are so adjusted that the centre of gravity of the combination is at the centre of the axis, they will remain wherever placed (Fig. 65*c*).

By giving one of the weights an impulse, rotate them on the axis, (*a*) when the centre of gravity is at a distance from the axis, (*b*) when it coincides with axis. Compare the motions.

13. Equilibrium of a Body resting on a Horizontal Surface.

When a body rests in equilibrium on a horizontal surface it is acted upon by two forces.

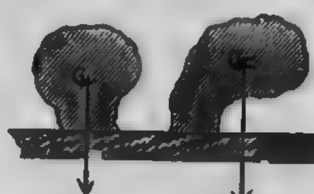


FIG. 66. FIG. 67.

(1) Its weight, which acts vertically downward through its centre of gravity.

(2) The resultant of the upward pressure at the points of contact.

Since this resultant must balance the weight, it must act vertically upward through the centre of gravity of the body (Figs. 66, 67).

Since the resultant of two like parallel forces always acts at a point between the forces (Art. 1, page 103), it follows that the resultant of the upward pressures at the points of contact must fall within the closed polygon formed by joining the extreme points of contact of the body and the plane (Fig. 66). This polygon is generally known as the **base** of the body, and the conditions of equilibrium are stated as follows:—

A rigid body under the action of gravity only, standing on a horizontal plane, is in equilibrium provided that the vertical line through the centre of gravity of the body cuts the plane at some point within its base.

14. To find experimentally the centre of gravity of a thin plane lamina.

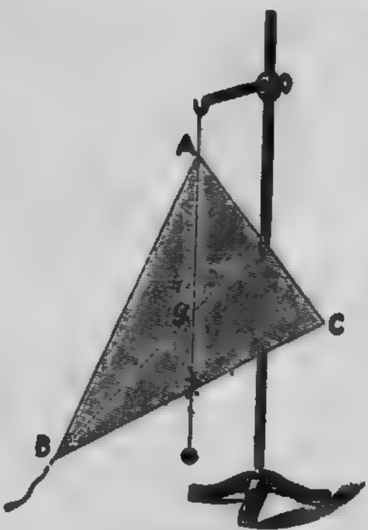


FIG. 68.

Attach a string to any point A (Fig. 68) of the body and suspend it by a string from a support.

By means of a plumb line suspended from the same point of support, draw a vertical line on the surface of

the lamina. The centre of gravity lies in this line. (Art. 11, page 148.)

Now suspend in the same way the lamina from another point B and draw another vertical line, and the centre of gravity of the lamina will be at *g*, the point of intersection of these lines. To verify this, the lamina may be suspended from other points and it will be found that the vertical lines through these points all pass through *g*.

EXERCISE XXVIII.

1. If a heavy uniform lamina, in the shape of an equilateral triangle, is suspended from any of its angles, show that the opposite side is always horizontal.

2. A triangular lamina is hung up by one of its angular points and when in equilibrium the opposite side is horizontal. Prove that the triangle is isosceles.

3. A system of three equal particles connected by rigid wires without weight forms a triangle, and when hung up by the middle point of one side rests with that side horizontal. Prove that the triangle is isosceles.

4. A piece of wire is bent to form three sides of a rectangle, and is then hung up by one of its angles. If the sides containing that angle be equally inclined to the horizon, show that the ratio of the arms will be $\sqrt{3}-1:1$.

5. An isosceles triangle is suspended (1) from the vertex, (2) from one of the equal angles. The angle between the two positions of the base is 60°. Find the angles of the triangle.

6. If a right-angled triangle is suspended from either of the points of trisection of the hypotenuse, show that it will rest with one side horizontal.

7. A right-angled isosceles triangle is suspended (1) from the vertex, (2) from one of the equal angles. Find the tangent of the angle between two positions of any one of the sides.

8. A triangular lamina when suspended from a point P in the side AB, rests with the side BC vertical. Show that AP=2BP.

9. The wheels of a hay cart are 10 feet apart and the centre of gravity of the cart and load is 12 feet above the ground and midway between the wheels. How much could either wheel be raised without the cart falling over?

10. How many coins of the same size, having the thickness $\frac{1}{20}$ the diameter, can stand in a cylindrical pile on an inclined plane of which the height is $\frac{1}{4}$ the base, if there is no slipping?

11. A number of cent pieces are cemented together so that each just laps over the one below it by the ninth part of its diameter. How many may be thus piled without falling?

12. A brick is laid with a quarter of its length projecting over the ridge of a wall; a brick and a quarter are laid on the first with a quarter of a brick [redacted] over the edge of the first brick; a brick and a half laid on this and so on. Prove that four such courses can be laid, but that if the fifth course is added the mass will topple over.

13. An isosceles triangle is placed with its base, which is 2 feet in length, upon a plane whose inclination is 30°, and is prevented from sliding by a small obstacle placed at the lowest point of the base. What is the greatest height which the triangle can have without toppling over?

14. A flat triangular board ABC, right-angled at A, stands with its plane vertical and its side AC on a horizontal plane; D is the middle point of AC. If the portion BAD is cut away, show that the board will be on the point of toppling over.

15. ABDC is a square, length of side 10 cm., standing on CD as a base in a vertical plane, and the triangle ACE is cut away. Find the least length of DE in order that the remainder should not fall.

16. A brick, $8 \times 3 \times 4$ inches in size, rests with its smallest face on an inclined plane, the 3 inch side being horizontal; the brick is prevented from sliding by friction. Find the greatest angle to which the plane can be raised without causing the brick to fall over.

17. A brick, whose dimensions are $8 \times 4 \times 3$, rests on a rough plane in such a way that it cannot slip, and the plane is tilted about a line parallel to the edge of the brick. Find the greatest and least angles of inclination for which the brick will just not upset.

18. A square table, whose mass is 10 kgm., stands on four legs placed respectively at the middle points of its sides. Find the greatest mass which can be put at one of the corners without upsetting the table.

19. A circular table, of mass 50 lbs., rests on three legs attached to three points in the circumference at equal distances apart. When the table rests on a horizontal plane what is the least mass which when placed on it will be on the point of upsetting it?

20. A square table, of mass 20 lbs., has legs at the middle points of its sides, and three equal masses, each 20 lbs., are placed at three angular points. What is the greatest mass that can be placed on the fourth corner so that equilibrium may be preserved?

21. The radius of the base of a cone is to its altitude as 2 : 15, the cone is placed on its base on a smooth inclined plane, and is kept from slipping by a string fastened to a point on the plane and to the rim of the base. Find the greatest inclination of the plane consistent with equilibrium. (Centre of gravity of a cone is in the line joining the vertex with the centre of the base at one-fourth of its length from the base.)

22. ABC is an isosceles triangle, of mass W, of which the angle A is 120°, and the side AB rests on a smooth horizontal table, the plane of the triangle being vertical. If a mass $\frac{W}{3}$ be hung at C, show that the triangle will be just on the point of toppling over.

23. A uniform triangular lamina ABC lies on a horizontal table, with the side BC on the table, and parallel to the edge, and one-ninth of the area of the triangle overhangs the table. Show that if a mass be placed at A greater than the mass of the triangle itself, the triangle will upset.

24. The side CD of a uniform square plate ABCD, whose mass is W, is bisected in E and the triangle AED is cut out. The plate ABCE is placed in a vertical position, with the side CE on a horizontal plane. What is the greatest mass that can be placed at A consistent with equilibrium?

25. A five-sided figure consisting of a square ABCD, with an isosceles triangle upon the side BC as base, is cut out of one piece of board. Find the greatest height of the triangle, that the figure may stand in a vertical position, with its side DC on a horizontal plane, without tumbling over.

CHAPTER XII.

FRICTION.

1. Friction.

We have, in the problems so far considered, assumed that the surfaces of two bodies in contact were perfectly smooth, and that consequently the stress between the bodies was at right angles to the surfaces in contact. A perfectly smooth surface, like a perfectly straight line or a perfectly fluid body, is but an ideal conception.

Our observations teach us that, while the surfaces of bodies differ widely in smoothness, it is possible to apply to any body resting upon another a small force parallel to the surfaces in contact without producing motion. This shows the existence of a balancing force.

The stress, therefore, between two bodies resting in contact along a plane surface is not necessarily at right angles to this plane, but may be resolved into two rectangular components, one, called the **normal pressure**, acting at right angles to this plane; the other, called **friction**, acting parallel to it.

2. [redacted] of Friction.

The direction of friction acting upon a body is opposite to that in which motion would take place if there were no friction.

When there is equilibrium, the magnitude of friction is equivalent to the magnitude of the least possible force required to maintain equilibrium.

EXERCISE XXIX.

1. A body rests on a horizontal plane, and is acted on by a force of 12 pounds, making an angle of 60° with the plane. What is the magnitude of the friction called into play?

2. A body rests on a horizontal plane, and is acted upon by a force of 20 pounds, making an angle of 30° with the plane. What is the magnitude of the friction called into play?

3. A block of wood is in equilibrium on a rough horizontal table, when a force of 3 grams acts due north, and a force of 4 grams acts due east on it. Find the magnitude of the friction exerted.

4. A body is in equilibrium on a rough horizontal plane, when forces of 7 pounds and 8 pounds act upon it. If the forces are parallel to the plane, and make an angle of 60° with each other, find the magnitude of the friction exerted. $R = \sqrt{P^2 + Q^2 + 2PQ\, c\ldots}$

5. A mass of 60 lbs. rests on a rough plane inclined to the horizon at an angle of (1) 30°, (2) 45°, (3) 60° to the horizon. Find the magnitude of the friction exerted.

6. A mass of 40 lbs. rests on a rough plane inclined to the horizon at an angle of 30°, and a force of 12 pounds acts upon the body parallel to the plane (1) upward, (2) downward. What is the amount of friction called into play?

Diff = 10 down

7. A mass of 100 lbs. rests on a rough plane inclined at an angle of 30° to the horizon, and two men push against it—one up the plane, with a force of 50 lbs.; the other down the plane, with a force of 60 pounds. What is the amount of friction exerted?

$100 \cos 60 + 10 = 60$

8. A mass of 100 lbs. rests on a rough plane inclined at an angle of 60° to the horizon, and is acted upon by a horizontal force of 10 pounds (1) toward the plane, (2) away from the plane. Find the magnitude of the friction exerted.

9. A mass of 10 lbs. is in equilibrium on a rough inclined plane of 1 foot in 4 feet. Find the friction exerted when the force of 5 pounds is applied to the mass (1) up the plane, (2) down the plane, (3) horizontally toward the plane.

3. Limiting Friction.

Experiment 1.

Arrange apparatus as shown in Fig. 69. The hardwood board AB is about 50 cm. in length and 20 cm. in width, and is provided with some device for holding it at any angle when turned on its hinges (Fig. 69). The upper surface of the board is made as smooth as possible. A vertical scale CD is placed at a given distance AC, say 30 cm., from A.

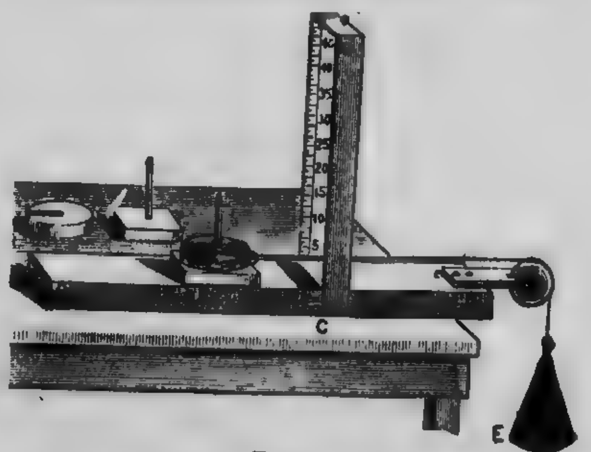


FIG. 69.

Several small pieces of board of different sizes should be provided. These should be cut from a hardwood board, one surface of which has been made as uniformly smooth as possible. Pieces 15 × 5 cm., 15 × 10 cm., 15 × 15 cm. are convenient sizes. Into each a vertical rod for holding weights in position is fastened, and an eye or hook to which a cord can be attached is screwed. It will be found convenient to make them all of the same weight, say one pound, by running lead into holes in their upper surfaces.

Rest the board AB in a horizontal position, and lay on it one of the small pieces of board. Upon this place any weight, say four pounds, and attach the cord and scale pan as shown in the figure. Place a small weight in the scale pan and add

others in succession until motion takes place. Gently tap the board each time a weight is added to the scale pan.

It will be found that by adding weights the tension of the string may be increased to a certain limit without producing motion, but if that limit is exceeded the board begins to move.

When a body is just on the point of sliding along another, the body is said to be in limiting equilibrium, and the friction exerted is called limiting friction.

4. Laws of Limiting Friction.

Experiment 2.

Arrange the apparatus used in Experiment 1, Fig. 69. Lay one of the small boards on AB and on it place a weight. Note the total weight supported by the board, including the weight of the board itself. This will then be the measure of the normal stress between the boards. Now add weights to the scale pan until the board just begins to move. Note the total weight suspended. Repeat the experiment several times, placing different weights on the board.

Denoting the normal pressures by R_1, R_2, R_3, etc., and the limiting frictions by F_1, F_2, F_3, etc., fill up the following table from your observations :—

NORMAL PRESSURES.	LIMITING FRICTIONS.	QUOTIENTS.
$R_1 =$	$F_1 =$	$\frac{F_1}{R_1} =$
$R_2 =$	$F_2 =$	$\frac{F_2}{R_2} =$
$R_3 =$	$F_3 =$	$\frac{F_3}{R_3} =$
Etc.	Etc.	Etc.

If the experiment is carefully performed the quotients

$$\frac{F_1}{R_1}, \quad \frac{F_2}{R_2}, \quad \frac{F_3}{R_3}$$

will be found to be approximately equal; that is, the ratio of the limiting friction to the normal pressure between the boards is constant.

Repeat the experiments, using the other pieces of board and making the normal pressures as before, R_1, R_2, R_3, etc. It will be found that, although the shapes and the areas of the surfaces in contact are different, the limiting frictions are approximately F_1, F_2, F_3, etc.

The limiting friction, therefore, is independent of the shape and the area of the surfaces of the boards in contact when the normal pressure remains unaltered.

The experiments of Coulomb and Morin, who have investigated this subject, show that the relations inferred from the above experiments hold approximately for different bodies in contact. The laws must not be looked upon as the statements of absolute truths, but rather as more or less accurate expressions of results determined by careful experiments. They may be thus stated :—

Law I.—**The ratio of the limiting friction to the normal pressure is constant when the substances in contact are unaltered.**

Law II.—**The limiting friction is independent of the area and the shape of the surfaces in contact when the normal pressure and the substances in contact remain unaltered.**

To which is sometimes added the following law of kinetic friction, the verification of which does not lie within the province of this work.

Law III.—**When motion takes place by one body sliding over another, the direction of friction is opposite to the direction of motion; the magnitude of the friction is independent of the velocity, but the ratio of the friction to the normal pressure is slightly less than the constant ratio of the limiting friction to the normal force when the bodies are at rest.**

5. Coefficient of Friction.

The constant ratio of the limiting friction to the normal pressure for a particular pair of substances in contact is called the **Coefficient of Friction**. It is generally denoted by μ.

Hence, if F denotes the limiting friction, and R the normal pressure,

$$\frac{F}{R} = \mu \quad \text{ma.}$$

$$\text{or } F = \mu R.$$

The values of μ differ widely for different substances, depending on the nature of the substances and the degree of polish of their surfaces.

One method of determining the coefficient of friction is given in Experiment 2, page 158. The following is another method of determining this coefficient, and of verifying the laws of limiting friction. If a body is supported upon an inclined plane and the equilibrium is limiting, the forces which keep it at rest are (Fig. 70)

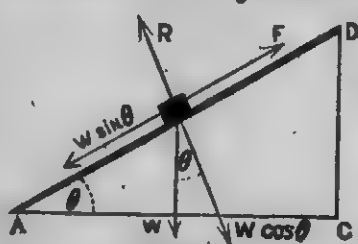


Fig. 70.

Its weight (W), acting vertically downward.

The limiting friction (F), acting along the plane, upward.

The normal pressure (R), acting at right angles to the plane, upward.

If θ denotes the angle DAC,

The resolved part of W along the plane $= W \sin \theta$,

And the resolved part of W at right angles to the plane $= W \cos \theta$. (Art. 3, page 78.)

Substituting for W its resolved parts along the plane and at right angles to it, the forces acting on the body are

(i) $W \sin \theta$, acting along the plane, downward.

(ii) F, acting along the plane, upward.

(iii) $W \cos \theta$, acting at right angles to the plane, downward.

(iv) R, acting at right angles to the plane, upward.

Since (i) and (ii) are at right angles to (iii) and (iv), and the body is in equilibrium,

(i) = (ii) and (iii) = (iv) Art. 6, page 83.

that is $F = W \sin \theta$

$R = W \cos \theta$,

therefore $\frac{F}{R} = \frac{W \sin \theta}{W \cos \theta} = \tan \theta =$ [illegible]

But $\frac{DC}{AC}$ may be determined in the following manner with the apparatus used in Experiments 1 and 2.

Experiment 3.

Place one of the small boards on AB and lay a weight upon the board. Gradually raise the end of AB turning it upon the hinges, and at the same time tap it gently. When the small board just begins to slide, fasten AB in position and observe the height CD indicated by the vertical scale. (Fig. 71.)

FIG. 71.

Thus $\frac{DC}{AC}$ is determined

and since $\frac{DC}{AC} = \tan\theta = \frac{F}{R}$,

$\frac{F}{R}$, or μ, the coefficient of friction of the board is determined.

Repeat the experiment, changing (1) the small boards, (2) the weights, (3) both boards and weights.

1. Does the ratio $\frac{F}{R}$ remain constant?

2. How does this experiment verify the laws of limiting friction?

6. Limiting Angle of Friction.

The angle which the direction of the resultant of the normal pressure and the limiting friction makes with the

direction of the normal pressure is called **the limiting angle of friction, or angle of repose.**

For example, if when a body rests upon an inclined plane, R is the normal pressure, F the limiting friction, and S the resultant of F and R, the angle α which S makes with R is the angle of friction. (Fig. 72.)

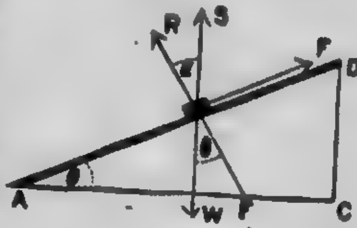


FIG. 72.

Since S is the resultant of F and R

$$S \sin \alpha = F$$

and $$S \cos \alpha = R \qquad \text{Art. 3, page 78.}$$

therefore $$\frac{S \sin \alpha}{S \cos \alpha} = \tan \alpha = \frac{F}{R}$$

But $$\frac{F}{R} = \tan \theta \qquad \text{Art. 5, page 161.}$$

therefore $$\tan \alpha = \tan \theta$$

or $$\alpha = \theta.$$

Hence, when a body rests upon an inclined plane under the action of gravity and the re-action of the plane only, and the equilibrium is limiting, **the angle of the inclination of the plane to the horizon is equal to the limiting angle of friction.**

The equality of the angles may be seen directly from the figure.

The body is in equilibrium under the action of the forces F, R and W.

The resultant of F and R is S, which is the re-action of the plane.

The body is, therefore, in equilibrium under the action of W and S.

Therefore W and S are equal and act in opposite directions in the same straight line.

Hence the angle α = the angle θ.

7. **Example.**

A mass of 12 lbs. rests on a rough plane inclined at an angle of 30° to the horizon. What force must be applied to it at an angle of 30° to the vertical that it may be on the point of moving up the plane, the coefficient of friction being $\frac{1}{2}$?

The forces acting on the body are (Fig. 73):—

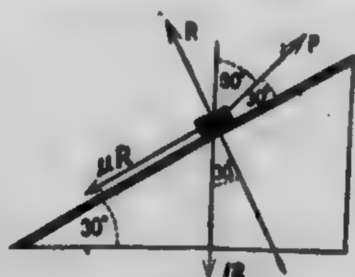


FIG. 73.

(i) Its weight, 12 pounds, acting vertically downward.

(ii) The normal pressure of the plane, R, acting at right angles to the plane, upward.

(iii) Friction, $F = \mu R = \frac{1}{2} R$, acting along the plane downward since the body is on the point of moving upward.

(iv) The required force, P, acting at an angle of 30° to the vertical = 30° to plane.

Resolve the forces along the plane, and at right angles to it.

Then, if X denotes the algebraic sum of the components along the plane, and Y the algebraic sum of those at right angles to it,

$$X = P\cos 30° - \tfrac{1}{2}R - 12\cos 60°$$

$$= \tfrac{1}{2}\sqrt{3}P - \tfrac{1}{2}R - 6$$

and

$$Y = P\cos 60° + R - 12\cos 30°$$

$$= \tfrac{1}{2}P + R - 6\sqrt{3}$$

Since the body is in equilibrium

$$X=0,\ Y=0 \qquad \text{Art. 6, page 52.}$$

therefore

$$\tfrac{1}{2}\sqrt{3}\,P-\tfrac{1}{2}R-6=0 \quad . \quad . \quad (1)$$

$$\tfrac{1}{2}P+R-6\sqrt{3}=0 \quad . \quad . \quad (2)$$

Eliminating R from (1) and (2) by taking (1) × 2 + (2)

$$P\left(\tfrac{1}{2}+\sqrt{3}\right)-12-6\sqrt{3}=0$$

or

$$P=\frac{24+12\sqrt{3}}{1+2\sqrt{3}}$$

3. Two bodies, of masses m_1 grams and m_2 grams, are connected by a light inextensible string; m_2 is placed on a rough plane inclined at an angle θ to the horizon, and the string

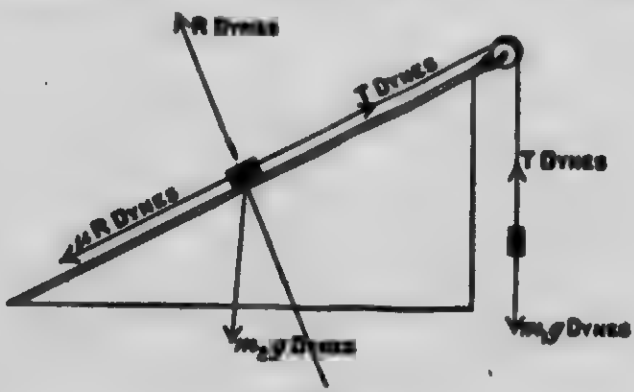


Fig. 74.

after passing over a small smooth pulley at the top of the plane (Fig. 74), supports m_1, which hangs vertically. If the coefficient of friction of the plane is μ, and m_1 descends, determine (1) the acceleration of the system, (2) the tension of the string.

Let T dynes be the tension of the string and a the acceleration of the masses. Now, considering the forces acting on m_1, we have, as in examples 1 and 2, pages 48 and 49,

$$T=m_1g-m_1a \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Consider the forces acting on m_2. These are :—

(a) Gravity, $m_2 g$ dynes, acting vertically downward.

(b) Normal pressure, acting at right angles to the plane upward. Let this be R dynes.

(c) Tension of the string, T dynes, acting along the plane upward.

(d) Friction, $F = \mu R$ dynes, acting along the plane downward (since the mass is moving upward).

Resolve these forces along the plane and at right angles to it.

Then if X denote the algebraic sum of the components along the plane, and Y the algebraic sum of those at right angles to it,

$$X = T - m_2 g \sin\theta - \mu R$$
$$Y = R - m_2 g \cos\theta.$$

Now, since m_2 has no acceleration perpendicular to the plane

$$R - m_2 g \cos\theta = 0$$

or

$$R = m_2 g \cos\theta.$$

The resultant force acting up the plane is

$$T - m_2 g \sin\theta - \mu R = T - m_2 g \sin\theta - \mu m_2 g \cos\theta,$$

since

$$R = m_2 g \cos\theta.$$

But the resultant force acting up the plane is also $m_2 a$.

Hence $T - m_2 g \sin\theta - \mu m_2 g \cos\theta = m_2 a$

or $T = m_2 c + m_2 g(\sin\theta + \mu\cos\theta)$ . . . . (2)

But $T = m_1 g - m_1 a$.

Hence $m_1 g - m_1 a = m_2 a + m_2 g(\sin\theta + \mu\cos\theta)$

or

$$a = \frac{m_1 - m_2(\sin\theta + \mu\cos\theta)}{m_1 + m_2} g \text{ cm. per sec. per sec.}$$

and

$$T = m_1 g - \frac{m_1 - m_2(\sin\theta + \mu\cos\theta)}{m_1 + m_2} m_1 g$$

$$= \frac{m_1 m_2(1 + \sin\theta + \mu\cos\theta)}{m_1 + m_2} g \text{ dynes.}$$

EXERCISE XXX.

1. A mass of 10 pounds rests on a rough horizontal plane. If the coefficient of friction is .2, find the least horizontal force which will move the mass. Find also the re-action of the plane.

2. A force of 5 pounds is the greatest horizontal force that can be applied to a mass of 75 pounds resting on a rough horizontal plane without moving it. What is the coefficient of friction?

3. A mass of 10 pounds is resting on a rough horizontal plane, and is acted on by a force which makes an angle of 45° with the plane. If the coefficient of friction is .5, find the force.

4. A body resting on a rough horizontal plane is on the point of moving when acted on by a force equal to its own weight inclined to the plane at an angle of 30°. Find the coefficient of friction.

5. A body placed on a rough plane is just on the point of sliding down when the plane is inclined to the horizon at an angle of (1) 60°, (2) 45°, (3) 30°. What is the coefficient of friction in each case?

6. A body placed on a rough inclined plane is on the point of sliding when the plane rises 3 feet in 6 feet. What is the coefficient of friction?

7. A mass of 20 lbs. rests on a rough plane inclined at an angle of 30° to the horizon. What force must be applied parallel to the plane that it may be on the point of moving up the plane, the coefficient of friction being .1?

8. A body, the mass of which is 30 lbs., rests on a rough inclined plane, the height of the plane being $\frac{3}{5}$ of its length. What force must be applied to the body parallel to the plane that it may be on the point of moving up the plane, the coefficient of friction being .75?

9. A rough plane is inclined to the horizon at an angle of 60°. What is the greatest mass which can be sustained upon it by a force of $10\sqrt{3}$ pounds acting parallel to the plane, if the coefficient of friction is $\sqrt{3}$? down

10. A mass of 14 lbs. when placed on a rough plane inclined to the horizon at an angle of 60° slides down unless a force of at least 7 pounds acts up the plane. What is the coefficient of friction?

11. A mass of 20 lbs. is on the point of moving up a rough plane inclined to the horizon at an angle of 45° when a horizontal force is applied to it. Find the horizontal force, if the coefficient of friction is .1.

12. A body, the mass of which is 4 lbs., rests in limiting equilibrium when the inclination of the plane to the horizon is 30°. Find the force which acting parallel to the plane will support the body when the inclination of the plane to the horizon is 60°.

13. A body placed on a rough plane inclined to the horizon at an angle of 30° is just on the point of moving upward when acted upon by a horizontal force equal to its own weight. Find the coefficient of friction.

14. If the smallest force which will move a mass of 3 lbs. along a horizontal plane is $\sqrt{3}$ pounds, find the greatest angle at which the plane may be inclined to the horizon before the mass begins to [redacted]

15. Show that in order to relieve a horse in drawing a sleigh the traces should be so placed as to make the angle of friction with the ground.

16. How would you place a brick on an inclined plane so that it would be the least likely to *tumble over?* Would it be less likely *to slide down* with one face in contact with the plane than another? Give reasons for your answer.

17. A particle, whose mass is m, slides down a rough plane inclined to the horizon at an angle of θ; if μ is the coefficient of friction, determine the acceleration.

18. A particle slides down a rough inclined plane whose inclination to the horizon is 45° and whose coefficient of friction is $\frac{1}{4}$. Show that the time of descending any space is twice what it would be if the plane were perfectly smooth.

CHAPTER XIII.

FLUID PRESSURE AT A POINT.

1. The Characteristic Properties which Distinguish the Solid State of Matter from the Fluid.

Experiment 1.

Take any solid body, such as a piece of wood or iron, lift it and place it on the table.

1. Does the whole move when a part moves?
2. Is its shape changed?
3. What is necessary to change its shape?

Experiment 2.

Put your fingers into a vessel containing water and try to lift the water out. With a spoon dip the water out of one vessel and place it in another of a different shape. Pour water on a horizontal surface. Try to grasp a handful of air.

1. Is the whole of the water lifted out when a part is raised?
2. Has it a definite shape of its own?
3. What shape does it take?
4. Can you lift a piece of air and carry it from one point to another? Has any portion of air a shape of its own?

Water and air belong to the class of bodies known as Fluids.

A solid is a body that possesses rigidity, that is the power to resist change of shape.

A fluid is a body which possesses no rigidity whatever, but which is deformed by the action of any force, however small.

2. The Characteristic Properties that Distinguish Liquid from Gaseous Fluids.

Experiment 3.

Take a glass tube (Fig. 75) closed at one end, fill it nearly full of water or any other liquid, insert a piston and push in on it.

Is there any change in the volume of the liquid?

FIG. 75.

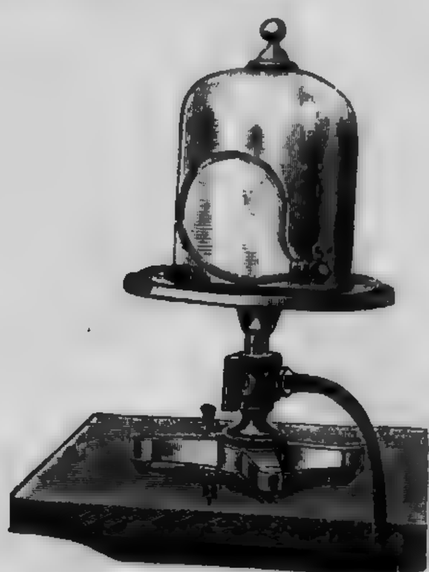


FIG. 76.

Experiment 4.

Repeat Experiment 3, having the tube filled with air instead of water.

1. What change takes place in the volume of the air?
2. What causes the change?

Experiment 5.

Place an elastic rubber balloon partially filled with air under the receiver of an air pump (Fig. 76). Exhaust the air from the receiver.

1. What change in the volume of the air in the balloon takes place?

2. How did removing the air from the receiver affect the pressure to which the balloon is subjected?

3. What caused the change in the volume of the air in the balloon?

On the basis of compressibility and expansibility, fluids are divided into two classes, **liquids** and **gases.**

A liquid is a highly incompressible fluid, that is, it is a body which possesses a definite volume but no definite shape, moulding itself into the shape of the containing vessel.

A gas is a compressible and expansible fluid, that is, it is a body which possesses neither definite shape nor definite volume, taking not only the shape but also the volume of the containing vessel.

3. Measure of the Fluid Pressure at a Point.

It is a matter of common experience that a fluid exerts a pressure upon the surfaces with which it is in contact. If a piston is inserted into the bottom (Fig. 77) or side (Fig. 78) of a vessel and the vessel filled with water a force must be exerted on the piston to prevent its being pushed out. If an upward pressure of 100 pounds is required to hold the piston in position, the pressure of the water on the surface of the piston must equal 100 pounds. Now if the area of the piston is 5 sq. inches, and the square inch is the unit of area, the pressure of the water on each unit-area of the surface of the piston is $\frac{100}{5}=20$ pounds. The pressure at every

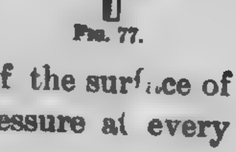


FIG. 77.

point in the surface is said to be 20 pounds per square inch.

Hence, the pressure of a fluid at a point in a plane-area is measured when the pressure is uniform over the plane by the force exerted on each unit of area.

When the piston is inserted into the side of a vessel filled with water (Fig. 78), the pressure of the water on the surface of the piston differs at different points, but is measured at any point P in the surface by the force which the water would exert on a unit-area if the pressure on this unit-area were uniform, and the same as at P, or if **the pressure over the plane is variable, the pressure at a point is measured by the force which would be exerted on a unit-area if the pressure were exerted over the whole unit-area at the same rate as at the point.**

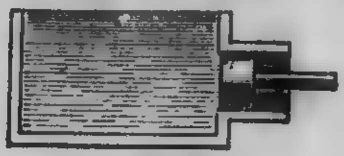


FIG. 78.

It should be carefully noted that the statement that the pressure at a point is 20 pounds per sq. inch does not imply that the area of the point is one square inch, or that the pressure upon the point is actually 20 pounds. The fact is that both the area of the point and the pressure upon it are infinitely small. When it is stated that the velocity of a train at a certain instant is 20 miles per hour, it is not inferred that the instant is an hour, or that the distance the train moves in the instant is 20 miles. It means that at a point of time the train was moving **at the rate of 20 miles per hour**; that is, if the train were to continue to move for an hour, and if its velocity each instant during the hour were the same as at the given point of time, it would pass over 20 miles.

Similarly, pressure at a point is 20 pounds to the square inch means that at a certain point in a plane the pressure is **at the rate of 20 pounds to the square inch; that is, if a surface were one square inch, and the pressure at every point in it the same as at the given point, the pressure would be 20 pounds.**

4. Pressure at a Point Within the Mass of a Fluid.

To measure the pressure of a fluid at any point within its mass, imagine an indefinitely small rigid plane so placed as to contain the point (Fig. 79). The plane will in no way affect the pressure of the fluid because it introduces no new forces, nor destroys any of those already existing. Now conceive the fluid removed from one side of the plane; and instead of the pressure of the fluid on that side, suppose a force X to keep the plane in position, then the fluid pressure on the other side of the plane must equal X. If the area of surface pressed is a, the pressure at a point in it is $\frac{X}{a}$.

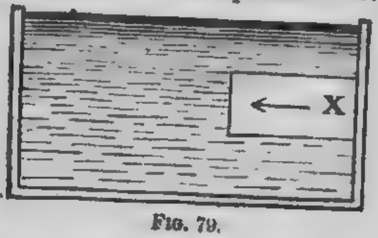


FIG. 79.

5. Laws of Fluid Pressure the Result of the Fundamental Properties of Fluids.

The following laws of pressure follow directly from fundamental properties of fluids.

Law I.—**The pressure at a point of a fluid at rest is perpendicular to any surface with which it is in contact.**

From its nature, the tangential resistance to change of shape in any fluid is zero when the fluid is at rest; therefore the strain between a fluid and the surface of a body in contact with it must be perpendicular to that surface.

This may be indirectly illustrated as follows:—

If possible at the point A in the side of a vessel (Fig. 80), let the pressure R of the fluid be not perpendicular. Resolve R into two forces, P and Q, P acting along the side of the vessel, and Q at right angles to it. (Art. 1, page 76.) The effect of Q is balanced by the re-action of the side of the vessel; but since P is unopposed a sliding motion of the particles of the fluid must be taking place in the direction AB.

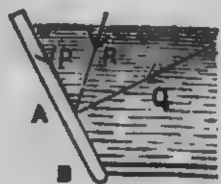


FIG. 80.

This is impossible, because by hypothesis the fluid is at rest; therefore the pressure of the fluid at A does not act in the direction RA. In the same way it can be shown that it does not act in any direction except in one perpendicular to the surface.

Law II.—**Law of transmission of Pressure—Pascal's Theory. Pressure exerted anywhere upon a mass of fluid is transmitted undiminished in all directions, and acts with equal intensity upon all equal surfaces, and in directions at right angles to these surfaces.**

It may be experimentally verified thus:—

Take a vessel ABC of any shape (Fig. 81), fill it with any fluid and insert pistons A, B and C; then it will be found that if the piston A is pressed by a force P, to keep them in position the pistons B and C must be pressed by forces which have the same ratios to P that the areas of the pistons B and C respectively have to the area of the piston A.

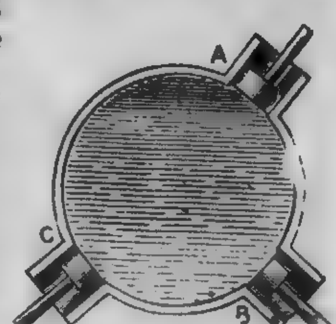


FIG. 81

6. Mechanical Application Hydrostatic Press.

The equal transmission of fluid pressure is the principle upon which all hydrostatic presses are constructed. Fig. 82 represents one of the simplest forms of these presses D and E are two hollow cylinders connected by a tube C, and partly filled with water; A and B are two pistons fitted into D and E respectively. Any force applied to A is transmitted through the fluid to B, and the pressures upon A and B are in the ratio of their areas. Thus, if the area of A is one square inch when that of B is ten square inches, a weight of one pound placed upon A will sustain a weight of ten pounds placed upon B.

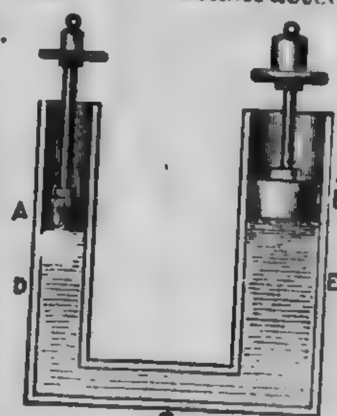


FIG. 82.

7. Hydrostatic Paradox.

By decreasing the area of A indefinitely and increasing that of B indefinitely, any force however small applied to A may, by the transmission of pressure through the fluid, be made to support upon B any weight however large. This is sometimes called a "**Hydrostatic Paradox.**"

LAW III.--Pressure at any point of a fluid at rest is equal in all directions. It follows from the principle of transmission of fluid pressure **that the pressure at any point within a fluid mass is the same for all directions.**

Let the piston A (Fig. 83) contain some unit of area, for distinctness say one square inch, and let C be any

point within the mass of the fluid; imagine it to be the centre of a circular plane, area one square inch and diameter *mn*. If the piston is pushed inward with a force of P pounds, the pressure on every square inch of the surface of the vessel will be P pounds; and, for the same reason, each face of the circular plane which contains C will be subjected to a force of P pounds. It is evident that on account of the uniform nature of the fluid the magnitude of these forces will remain unchanged if the circular plane is turned round to take different positions in the fluid mass; therefore the pressure at the point which it contains must be the same for all directions.

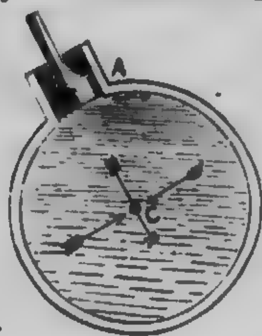


FIG. 83.

EXERCISE XXXI.

1. A fluid pressure of 1,728 pounds is uniformly distributed over a surface whose area is 3 sq. ft. Find the measure of the pressure at a point in the surface (1) when the unit-area is 1 sq. in., (2) when it is 1 sq. ft., the unit of force in each case being 1 pound.

2. The pressure is uniform over the whole of a sq. yard of a plane-area in contact with a fluid, and is 7,776 pounds. Find the measure of the pressure at a point (1) when the unit of length is 1 in., (2) when it is 3 in., the unit of force in each case being the pound.

3. The uniform pressure of a fluid over a circular plane, diameter 14 cm., is 770 kgm.; find the measure of the pressure at a point (1) when the unit-area is 1 sq. mm., (2) when it is 1 sq. dcm.; if the unit of force is the gram.

4. A rectangular surface, length 50 cm. and width 4 cm., is subjected to a uniformly distributed fluid pressure of 4 kgm. Find the measure of the pressure at a point (1) when the unit of length

is 1 mm., (2) when the unit of length is 2 mm., if the unit of force is the gram.

5. If the area of a piston inserted in a closed vessel is $3\frac{1}{4}$ sq. in., and if it is pressed with a force of 35 pounds, find the pressure which it will transmit to a surface of $7\frac{1}{2}$ sq. inches.

6. A closed vessel is filled with fluid and two circular pistons whose diameters are respectively 3 in. and 7 in. inserted; if the pressure on the larger piston is *a* pounds, find the pressure on the smaller.

7. The horizontal cross-section of the neck of a glass bottle, just capable of sustaining a pressure of 11 pounds to the sq. in., is $2\frac{3}{4}$ sq. in. It is filled with a fluid supposed weightless, and a piston is inserted into the neck. What is the least force that must be applied to the piston to break the bottle?

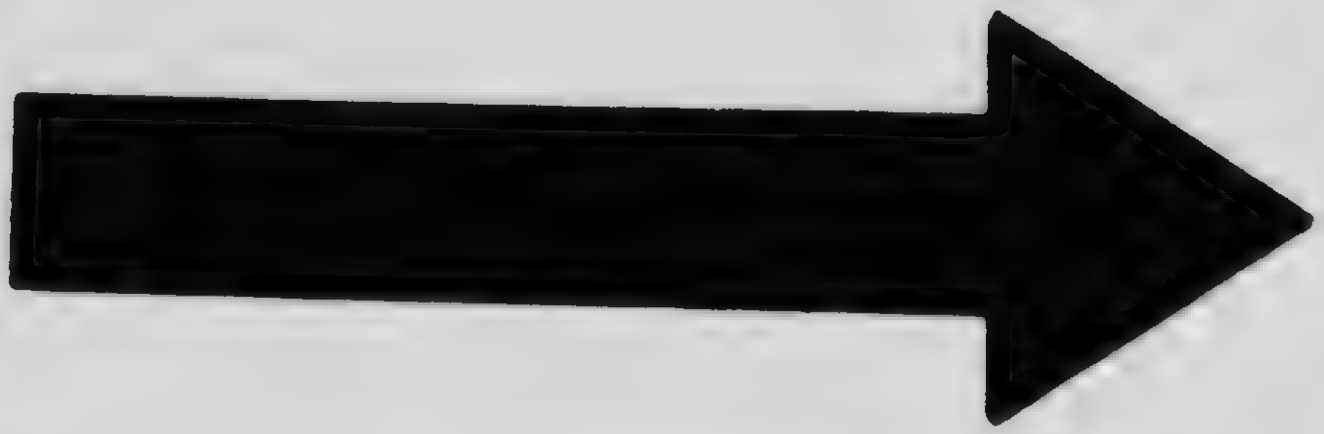
8. If the diameter of the small piston (Fig. 82) is 5 cm., and that of the larger one 2.5 metres, and if the small piston is pressed with a force of 8 gms., what force will it transmit to the large piston?

9. In the same machine the horizontal cross-section of the small piston is 3 sq. cm.; with what force must it be pressed that it may sustain a force of 7.25 kgm. applied to a piston whose horizontal cross-section is 7 sq. dcm.?

10. If the area of the small piston is 2.5 sq. dcm., and if it is pressed with a force of 6.25 centigrams, find the area of the large piston when a pressure of 3.75 grams is transmitted to it.

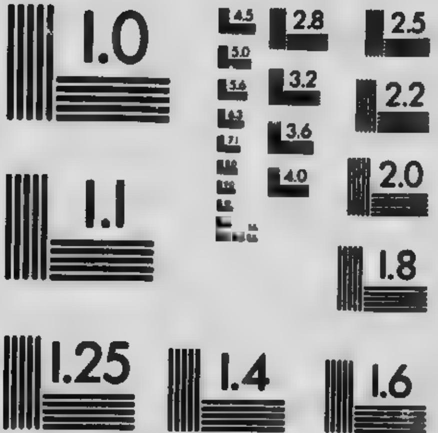
11. A pressure of 5 tons on the large piston, diameter $2\frac{1}{4}$ feet, transmits a pressure of 2.25 pounds to the small piston. Find the diameter of the small piston.

12. A rectangular box is divided into two compartments by a partition just capable of sustaining a pressure of 1 gram on 1 sq. mm. Both compartments are filled with a fluid supposed weightless, and a piston, area 2 sq. mm. is inserted into the first compartment, and another, area 350 sq. cm., is inserted into the second compartment. If a pressure of 2.5 centigrams is applied to the first piston, what pressure must be applied simultaneously to the second to break the partition?



MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



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13. A piston, area 4½ sq. in., is inserted into a rectangular box whose internal dimensions are, length 6 ft., width 3 ft., depth 3 ft. 4 in. If the vessel is filled with water and the piston pressed with a force of 10 oz., find the total pressure on the inside of the box due to the pressure on the piston.

14. A cubical box whose edge is 6 cm. in length is closed by a horizontal lid, and filled with a fluid supposed weightless. An opening of 2 sq. mm. in area is made in the lid, and a piston whose weight is 4 grams is inserted. Find the least weight which must be placed upon the lid to keep it down, if the weight of the lid is 196 grams.

CHAPTER XIV.

EQUILIBRIUM OF FLUIDS UNDER THE ACTION OF GRAVITY.

The forces so far represented as acting on fluids have been pressures applied to their surfaces, and the principles of fluid pressure already investigated are the result of the peculiar constitution of fluids, and are independent of the action of gravity. We shall now establish certain propositions which result from the action of gravity on fluids.

1. Pressures at Points in the same Horizontal Plane.

Experiment 1.

Cut the funnel-shaped end from a thistle tube, leaving about half an inch of the stem connected with it. Cement across the mouth of the funnel a piece of thin sheet rubber. This may be done by warming the lip of the funnel and pressing it down first on a cake of beeswax resin cement and then on the sheet rubber. Procure a U-shaped tube, which for convenience in emptying, should have an offset at the bottom closed with a rubber tube and pinch-cock. Connect the U-tube by means of a short rubber tube and a long bent glass tube with the funnel, and support the whole at such a height above the table that a tall glass jar can be slid under the mouth of the funnel (Fig. 84).

Partially fill the U-tube with water and press the rubber membrane with the finger.

1. What change takes place in the position of the water in the tube? Why?

2. How is (*a*) an increase, (*b*) a decrease in the pressure on the membrane indicated by the water in the tube?

3. How does the tube when filled act as a pressure-gauge?

Allow the water to run out of the pressure-gauge. Fill the jar with water, place it under the funnel and raise it up until

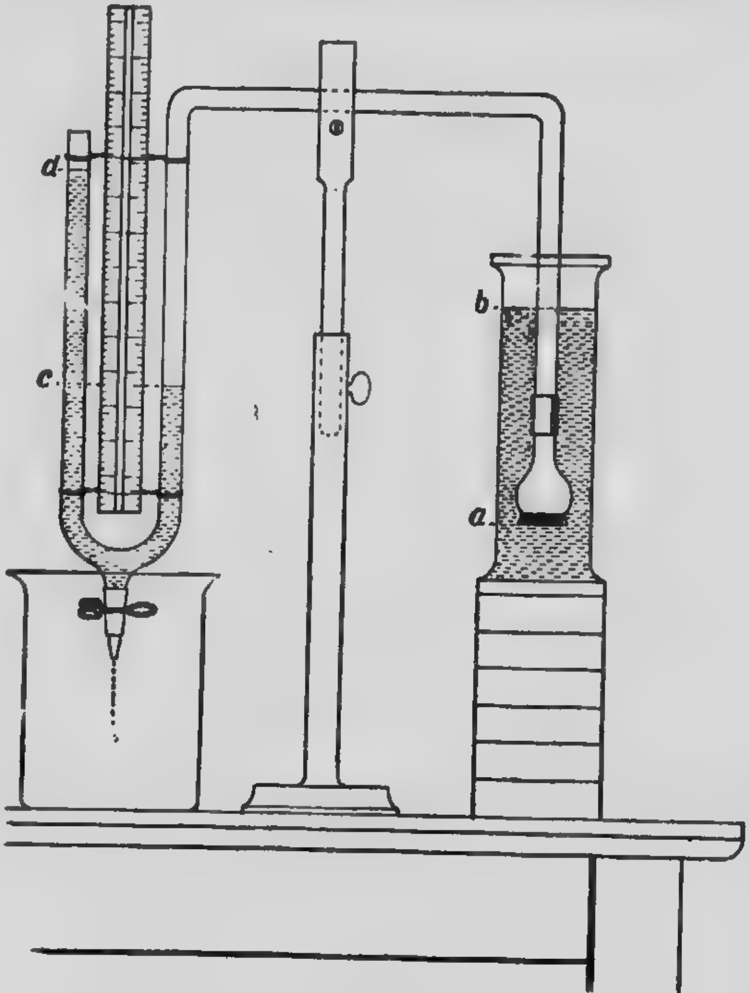

FIG. 84.

its bottom is near the mouth of the funnel. Observe the shape of the rubber membrane.

What is the cause of the change in shape?

Pour water into the pressure-gauge until the membrane resumes its original plane contour.

Move the jar from side to side, keeping the membrane in the same horizontal plane. Does the pressure-gauge indicate any difference in pressure for different positions of the jar?

The experiment tends to show that the pressure of a liquid at rest under gravity is the same at all points of the same horizontal plane.

This law may be shown to follow directly from the act of gravity on liquids as follows:—

Take any two points A and B (Fig. 85) in the same horizontal plane; consider a very thin cylinder of fluid whose axis is AB.

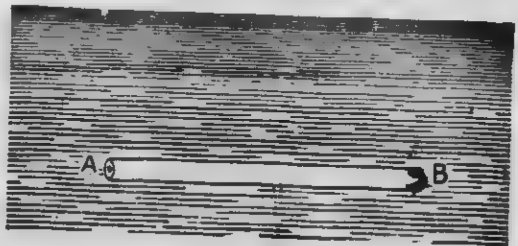

Fig. 85.

The cylinder is kept at rest by

(i) The fluid pressures on the curved surfaces, perpendicular to the axis.

(ii) The weight of the cylinder, acting vertically downward and hence perpendicular to the axis.

(iii) The fluid pressures on the ends A and B, perpendicular to these ends.

Since (i) and (ii) have no tendency to move the cylinder in a horizontal direction, and since the cylinder is at rest, the pressure on the end A must equal the pressure on the end B.

Now if p denotes the measure of the pressure at a point in the end A, and p_1 the measure of the pressure at

a point in the end B, and if a is the area of each end taken very small that the pressure on each end may be very nearly uniform and of the same intensity as at the middle point, therefore the pressure at the end A is pa and that at the end B is p_1a; but these pressures are equal, therefore

$$pa = p_1a$$

or

$$p = p_1$$

2. Relation between Pressure and Depth.

Experiment 2.

Arrange apparatus as in Experiment 1, page 179, fill the jar with water, place it under the funnel and raise it up until its bottom is near the mouth of the funnel. Support it in this position upon a series of blocks. Pour water into the pressure-gauge until the membrane becomes horizontal.

Now measure with a scale, (1) the depth *ab* of the membrane below the surface of the water, (2) the difference in level *cd* of the water in the limbs of the pressure-gauge.

Remove the supporting blocks one at a time. After the removal of each block adjust the water level in the pressure-gauge and take the measurements noted above. Tabulate the results thus:

Depth.	Reading of Pressure-Gauge.

Plot a curve showing the relation of pressure to depth.

This experiment tends to prove that the pressure at any point of a liquid at rest under gravity varies as the depth. This law may be shown to follow directly from the action of gravity on a liquid.

Take any point A in the liquid (Fig. 86), and imagine AB drawn vertically to the surface. Consider a thin cylinder of liquid whose axis is AB.

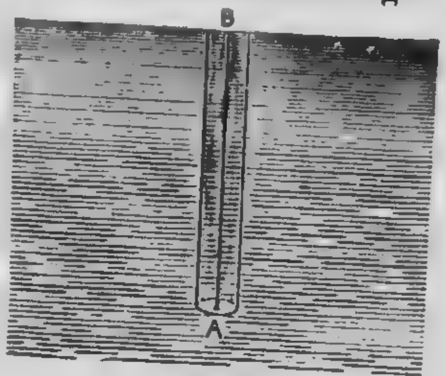

FIG. 86.

The cylinder is kept at rest by

(i) The fluid pressures on the curved surfaces of the cylinder, perpendicular to the axis.

(ii) The weight of the cylinder, acting vertically downward.

(iii) The fluid pressure on the end A, acting vertically upward.

Since (i) has no tendency to move the cylinder in a vertical direction, and since the cylinder is at rest, the pressure on the end A must equal the weight of the cylinder.

If p denotes the measure of the pressure at a point in the end A and a is the area of that end, the fluid pressure upon it is pa; and if z is the depth of the point, *i.e.*, the length of the cylinder, and ρ the weight of a unit volume of the liquid, the weight of the cylinder is ρaz; but the fluid pressure on the end A equals the weight of the cylinder,

therefore $$pa = \rho az$$

or $$p = \rho z$$

Since ρ is constant for the same liquid, any change in z will cause a corresponding change in p. Hence **the pressure at a point varies as the depth.**

For example, if the pressure at a point is 10 pound per sq. in. at a depth of 2 ft., it will be 20 pounds per sq. in. at a depth of 4 ft., 30 pounds per sq. in. at a depth of 6 ft., etc.

3. The Surface of a Liquid at Rest under Gravity.


Fig. 87.

Experiment 3.

Pour enough mercury into a bowl or a dinner plate to cover its bottom. Hold a plumb line over the surface of the mercury (Fig. 87).

1. What direction does a plumb-line always take?

2. What direction does the image of the line take with regard to the line itself?

3. What then must be the position of the surface of the mercury?

Experiment 4.

Pour water into a series of connecting tubes of various sizes and shapes. An apparatus for this purpose can be made by cutting the bottom off a glass bottle, inverting it and inserting tubes through a cork as shown in Fig. 88. Very small tubes should not be used.

1. Does the water reach the same level in each tube?

2. What would be the result if some very small tubes were used?

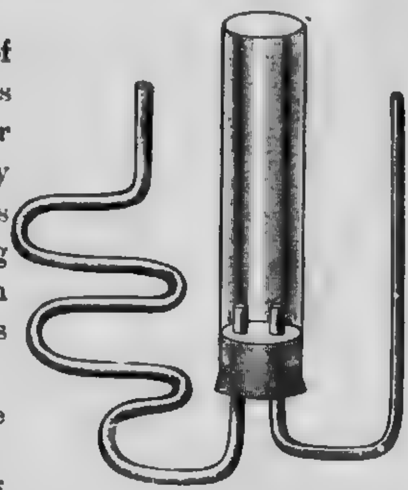
Fig. 88.

The surface of a liquid at rest is horizontal.

This law also may be shown to follow directly from the action of gravity on liquids.

Take any two points in a horizontal plane within the liquid, and imagine vertical lines AC, BD to be drawn to the surface (Fig. 89).

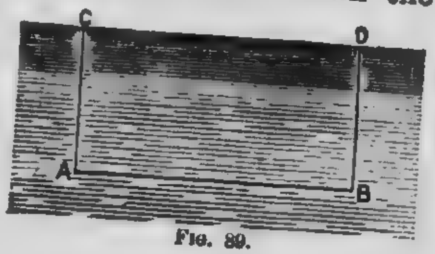

FIG. 89.

Then the pressure at A $=\rho \times AC$ and the pressure at B $= \rho \times BD$. (Art. 2); but the pressure at A = the pressure at B, because the points are in the same horizontal plane (Art. 1),

therefore $$\rho \times AC = \rho \times BD$$
or $$AC = BD$$

and since AC is equal to BD and is also practically parallel with it, CD is parallel to AB (Euclid I, 33). Hence C and D must be in the same horizontal plane.

In the case of a large sheet of water on the earth, the surface is not horizontal, but curved. The vertical lines AC and BD when at a great distance apart can no longer be looked upon as practically parallel.

4. Relation of Rate of Pressure to the Mass of the Liquid Pressing when the Depth is Constant.

Experiment 5.

Connect a U-shaped glass tube A with a collar B into which can be screwed tubes of different shapes, C, D, E, F (Fig. 90). Pour mercury into the U-tube until it reaches nearly up to the collar. Screw one of the tubes, say C, into the collar, and fill it with water up to any height indicated by a pointer P. Note the height of the mercury in the open limb of the U-tube. Now replace the tube C by each of the others in

succession, filling them to the same height, and noting the height of the mercury in the U-tube.

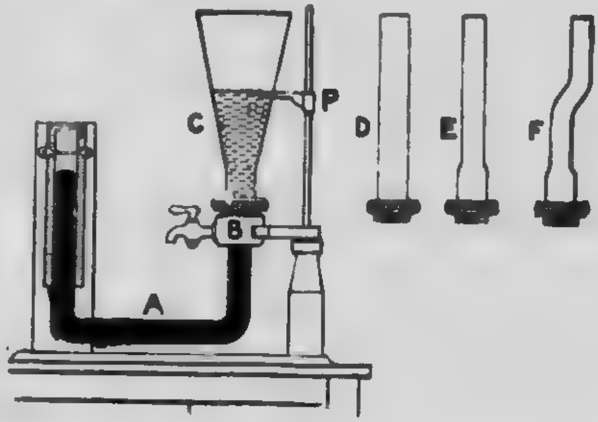

FIG. 90.

Compare for the different cases (*a*) the pressures on the surface of the mercury due to the water poured into the tubes, (*b*) the masses of the water pressing.

The above experiment tends to demonstrate the following law :—

The pressure in a given liquid is dependent only upon the depth. It is independent of the form of the vessel and of the amount of liquid which it contains.

EXERCISE XXXII.

1. If the pressure of a liquid at a depth of 14 ft. 3 in. is 6 pounds to the sq. in., find the pressure at a depth of 21 ft. 8 in.

2. If the pressure at a depth of 5.6 metres is 2.8 gm., what is the pressure at a depth of 7.5 cm. ?

3. If the pressure on a sq. in. at a depth of 40 cm. is 10 lbs., find the pressure 6 cm. lower down.

4. If the pressure on a sq. mm. at a depth of 5 metres is 2.5 gm., find the pressure 4 dcm. higher up.

5. In two uniform liquids the pressures are the same at depths of 3 and 4 metres respectively. Compare the pressures at depths of 12 and 18 metres respectively.

6. In two uniform liquids the pressures are the same at depths of 1 metre and 1 dcm. respectively. Compare the pressures at depths of 1 dcm. and 1 metre respectively.

7. In three uniform liquids the pressures are the same at depths of 2, 3, and 4 in. respectively. Compare the pressures at depths of 5, 3, and 6 in. respectively.

8. If the pressures at the same depth in two uniform liquids are as 5 : 6, compare the depths when the pressures are in the ratio 2 : 3.

9. Find the depth of a pool of water in which a stick 15 ft. long stands vertically upon the bottom, if the pressure at the top of the stick is to the pressure at the bottom of it in ratio 3 : 4.

10. The pressure at the bottom of a stick standing vertically in a pool of water 15 metres deep is to the pressure at the top in the ratio 5 : 2. Find the length of the stick.

11. Find the measure of the pressure at a point 72 ft. below the surface of a pool of water, when the unit of length is 1 in., the unit force 1 lb., and the density of water $62\frac{1}{2}$ lbs. per cubic foot.

12. A reservoir of water is 100 metres above the level of the ground floor of a house. Find the pressure of the water at a point in a water-pipe at a height of 10 metres above the ground floor when the unit-area is 1 sq. cm., and the weight of 1 c.cm. of water is 1 gram.

13. The pressure at a point within a body of water under the action of gravity is 100 lbs. per square inch. If the weight of a cu. ft. of water is 1,000 oz., find the depth of the point below the surface.

14. A reservoir is 200 ft. above the level of the ground floor of a house, and the pressure of the water at a point in a faucet in an upper room of the house is $73\frac{113}{144}$ pounds per square inch. Find the height of the faucet above the ground floor if 1 cu. foot of water weighs 1,000 oz.

15. Find in grams per square centimetre the pressure at a point due to a column of mercury 1 metre high, the density of mercury being 13.6 grams per cubic centimetre.

16. What must be the height of a column of mercury to exert a pressure of 1 kilogram per square centimetre?

17. The density of sea water is 1.025 grams per cubic centimetre. Calculate the pressure in grams per square centimetre at a depth of 40 metres below the surface of the sea.

18. A spherical boiler 4 feet in height is half full of water and half full of steam. What is the difference, in pounds per square foot, between the pressure at a point at the bottom and at the top of the boiler? (A cubic foot of water weighs $62\frac{1}{2}$ pounds.)

CHAPTER XV.

WHOLE PRESSURE.

We have seen that when any surface is in contact with a fluid, the fluid exerts a pressure at each point of the surface perpendicular to it. The sum of all these pressures is called the **whole pressure**[1] on the surface immersed.

1. To Determine the Measure of the Whole Pressure on a Plane Surface.

Suppose the surface to be divided into a number of elements so small that the pressure may be regarded as uniform over each. Let a be the area of one of these elements and z be its depth below the surface.

The pressure on this area $a = \rho a z$ when ρ is the weight of a unit volume of the liquid.

Now since the surface is plane the pressures on the various elements are all parallel. The whole pressure is, therefore, the sum of the pressures on the elements.

Then if $a_1, a_2 \quad . \quad . \quad . \quad a_n$ are the areas of the elements, and $z_1, z_2 \quad . \quad . \quad . \quad z_n$ are their depths, and P is the whole pressure, $\mathrm{P} = \rho\,(a_1 z_1 + a_2 z_2 + \quad . \quad . \quad . \quad + a_n z_n)$.

But if $\bar{z}$ is the depth of the centre of gravity of a number of particles $a_1, a_2 \quad . \quad . \quad . \quad a_n$.

$$\bar{z} = \frac{a_1 z_1 + a_2 z_2 + \quad . \quad . \quad . \quad + a_n z_n}{a_1 + a_2 + \quad . \quad . \quad . \quad + a_n}$$

(Art. 10, page 145)

or $a_1 z_1 + a_2 z_2 + \quad . \quad . \quad . \quad + a_n z_n = \bar{z}\,(a_1 + a_2 + \quad . \quad . \quad . + a_n)$.

Therefore, $\mathrm{P} = \rho \bar{z}\,(a_1 + a_2 + \quad . \quad . \quad . \quad + a_n)$.

[1] The terms **pressure-resultant** and **resultant thrust** are by some authors used to denote the particular force referred to, when the term pressure is restricted to denote rate of pressure, or force per unit area.

But $a_1 + a_2 + \quad . \quad . \quad . \quad . \quad + a_n$ is the area of the plane surface. Hence, $P = \rho z A$, where A is the area of the surface.

Hence,

The whole pressure on a plane = area of the plane × depth of its centre of gravity below the surface of liquid × density of liquid.

2. **Examples.**

1. Find the whole pressure on a rectangular surface 8 ft. by 6 ft., immersed vertically in water with the longer side parallel to, and 2 ft. below the surface of the water.

Area of the surface pressed $= 8 \times 6 = 48$ sq. ft.

The centre of gravity of the surface is 3 ft. below the upper horizontal side, or 5 feet below the surface of the water.

Then the volume of the column of water pressing on the surface $= 48 \times 5$ cubic ft.; and, since 1 cubic ft. of water weighs $62\frac{1}{2}$ pounds, the total pressure on the surface $= 48 \times 5 \times 62\frac{1}{2} = 15{,}000$ pounds.

2. What is the whole pressure exerted against a mill-dam whose length is 100 ft., the part submerged being 10 ft. wide and the water being 6 feet deep?

Area of part submerged $= 100 \times 10 = 1{,}000$ sq. ft.

Depth of the centre of gravity of the part submerged from the surface of the water $= \frac{1}{2}$ the depth of the water $= 3$ ft.

Then the volume of the column of water pressing on the part submerged $= 1{,}000 \times 3 = 3{,}000$ cubic ft. Therefore, the whole pressure $= 3{,}000 \times 62\frac{1}{2} = 187{,}500$ pounds.

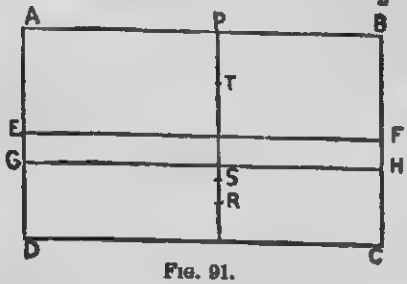

Fig. 91.

3. The flood-gate of a canal is 20 ft. wide and 12 ft. deep, and is placed vertically in the canal, the water being on one side only, and level with the upper edge of the gate; find the whole pressure on (i) the upper one-half of the gate, (ii) the lower one-half of the gate, (iii) the lowest one-third of the gate.

Let ABCD represent the gate (Fig. 91),

ABFE " upper one-half,

EFCD " lower "

GHCD " lowest one-third,

and let T, S and R denote the centres of gravity of upper one-half, lower one-half and lowest one-third respectively.

(i) Area of ABFE $= 20 \times 6 = 120$ sq. ft.

Distance PT of the centre of gravity of ABFE below the surface of the water $= 3$ ft.

Therefore the whole pressure on ABFE $= 120 \times 3 \times 62\frac{1}{2}$
$= 22{,}500$ pounds.

(ii) Area of EFCD $= 20 \times 6 = 120$ sq. ft.

Distance PS of the centre of gravity of EFCD below the surface of the water $= 9$ ft.

Therefore the whole pressure on EFCD $= 120 \times 9 \times 62\frac{1}{2}$
$= 67{,}500$ pounds.

(iii) Area of GHCD $= 20 \times 4 = 80$ sq. ft.

Distance PR of the centre of gravity of GHCD below the surface of the water $= 10$ ft.

Therefore the whole pressure on GHCD $= 80 \times 10 \times 62\frac{1}{2}$
$= 50{,}000$ pounds.

EXERCISE XXXIII.

1. Find the whole pressure on a rectangular plane 2 ft. by 4 ft. when immersed in water so that its centre of gravity is 10 ft. below the surface of the water.

2. Find the whole pressure on a rectangular plane 2 metres by 3 metres, immersed horizontally to a depth of 5 metres in water.

3. A rectangular plane 4 ft. by 3 ft. is immersed vertically in water with its shorter sides horizontal, the upper one being 3 ft. below the surface of the water. Find the whole pressure on the surface.

1000 × 500 × 2.15
1000 10750

4. A rectangular plane 10 metres by 5 metres is immersed in water with one of its sides horizontal, the upper being 2 metres and the lower 2.3 metres below the surface of the water. Find the whole pressure on it.

5. A circular surface whose radius is 7 feet, is immersed (1) horizontally, (2) vertically in water. If the depth of the centre of the circle in each case is 8 ft., find the pressure on the surface.

6. The water in a canal lock rises to a height of 10 ft. against one side of a vertical flood-gate whose breadth is 12 ft. Find the pressure against it.

7. Find the whole pressure against a mill dam 40 metres long and 2 metres wide when the water is level with the upper edge of the dam, the lower edge of the dam being 1.8 metres beneath the surface.

8. The water in a canal lock rises to a depth of 20 ft. against a vertical flood-gate whose width is 20 ft. Find the pressure on (1) the whole gate, (2) the upper half, (3) the lower half.

9. A rectangular surface 12 ft. by 14 ft. is immersed vertically in water with the longer sides horizontal, the upper being 8 feet below the surface. Find the pressure on (1) the whole surface, (2) the lowest one-quarter, (3) the upper two-thirds, (4) the lowest one-third.

10. A rectangular surface 12 ft. by 8 ft. is immersed in water with its short sides horizontal, the upper being 2 ft. and the lower being 12 ft. below the surface. Find the pressure on (1) the whole surface, (2) the highest one-fifth, (3) the lowest two-fifths.

11. A mill-dam is 8 metres long and 3 metres wide. If the water is level with the top of the dam and the lower edge of the dam is 2 metres below the surface of the water, find the pressure on (1) the whole dam, (2) the upper one-half, (3) the lowest one-quarter.

12. The water in a canal lock rises to a height of 6 metres against one side, and to a height of 4 metres against the other side of a vertical flood-gate whose breadth is 7 metres. Find the whole pressure against the gate.

13. A cube of 1 foot edge is suspended in water with its upper face horizontal and at a depth of $2\frac{1}{2}$ feet below the surface. Find the pressure on each face of the cube.

14. A dyke to shut out the sea is 100 ft. long and is built in courses of masonry 2 ft. high. If the water rises against it to a height of 20 ft., find the pressure on (1) the 2nd course, (2) the 5th course, (3) the 7th course. The courses are to be counted from the bottom.

15. An equilateral triangular plate is immersed vertically in water with an edge 60 cm. in length on the surface. Find the pressure upon one side of it.

16. An isosceles triangular plate whose base is 8 ft. and each of whose equal edges is 5 feet, rests in water in a vertical position with its base horizontal and its vertex in the surface of the water. Find the pressure upon one side of it.

17. A vessel whose base is a square the side of which measures 6 in., contains mercury to the depth of an inch and water to the depth of $10\frac{1}{2}$ in. If the sp. gr. of the mercury is 13.5, find the pressure on the base of the vessel.

18. To what depth must a rectangular surface of 5 sq. ft. in area be sunk in water that it may sustain a pressure of 31,250 pounds?

19. To what depth can a piece of glass whose surface is 60 sq. cm. and just capable of sustaining a pressure of 48 kgm., be sunk in water before it breaks?

20. The cork of a bottle will just sustain a force of 125 pounds to the sq. inch. To what depth can the bottle be sunk before the cork is driven in?

21. Two squares whose sides are 4 and 6 respectively are immersed vertically in a liquid, one side of each being horizontal. The first square has its upper side at a depth of 10 in. below the surface. To what depth must the second square be sunk that the pressure on it may be the same as that upon the first?

22. A closed cubical box 12 inches high is filled with mercury of sp. gr. 13.5, and is placed on the flat bottom of a pool of water. What must be the depth of the pool in order that the whole

pressure from within on one of the vertical faces may equal the whole pressure from without, assuming that the external and internal faces are of the same area ?

23. How deep must a 3-inch cube be sunk in water with two of its faces horizontal that the whole pressure on five square inches of the bottom may be equal to that on 6 square inches of the top ? What then will be the whole pressure on each of the faces ?

24. A cube floats with a face level with the surface of a fluid. Find the ratio of the pressures against the bottom and one of the sides.

25. A rectangular box 2 cm. long, 1.5 cm. wide, and 8 mm. deep, is filled with water. Find the pressure on (1) the bottom, (2) a side, (3) an end.

26. Compare the pressures on the bottom and a side of a cubical vessel filled with any fluid. Why is the result the same as that of question 24 ?

27. A cubical vessel is filled with two liquids whose specific gravities are 1 and 0.8 respectively. They do not mix, and their volumes are equal. Find the ratio of the pressure on the upper to that on the lower half of one of the vertical faces of the cube.

28. A cylindrical vessel, height 200 cm. and radius of base 70 cm., is filled with water. Find the pressure on (1) the bottom, (2) the curved surface.

29. A cylindrical vessel with smooth internal surfaces 40 cm. high and 14 cm. in diameter, is filled with water and closed by a piston weighing   kgm. Find the ~~pressure on (1) the bottom~~, (2) the curved surface.

30. A vessel is in the shape of a pyramid which is 4 ft. high and has a square base, each edge of which is 6 ft. Find the pressure on ~~(1) the base~~, (2) a side, when it is filled with water. If the vessel is supposed weightless, find the pressure on the table upon which the vessel stands.

31. A conical vessel, supposed weightless, 8 cm. high and radius of base 6 cm., is filled with water. Find the pressure on (1) the base, (2) the curved surface, (3) the table upon which the vessel stands.

32. A piston, 2 sq. mm. in area, and weighing 10 grams, is inserted into the upper side of a closed cubical box, each edge of which measures 8 cm. If the box is filled with water, find the whole pressure on the entire internal surface of the box.

33. A cylindrical vessel, 1.2 metres high, and 1.4 metres in diameter, is filled with water. Find the pressure on (1) ~~the base of cylinder~~, (2) the upper one-half of curved surface, (3) the lowest one-third of curved surface.

34. A conical vessel, with its vertex downward, is filled with water. Find the total pressure on its curved surface, the diameter of the base being 1 metre, and the height being 1.2 metres.

35. To what depth must a liquid be poured into a rectangular box, the base of which is a square whose side is 4, that the sum of the pressures on the sides may be four times the pressure on the base?

36. Find the height of a cylinder, the diameter of which is 12 in., that the pressure on the curved surface may be three times the pressure on the base, when the cylinder is filled with any liquid.

37. A tube whose internal cross-section is 1 sq. cm. opens freely into a water tank whose internal cross-section is 4 sq. m. What pressure must be exerted against a piston which works in the tube by the water rising in the tank to a height of 4 metres above the level of the piston?

38. A rectangular vessel 80 cm. long, 20 cm. wide, and 60 cm. deep, supposed weightless, is placed ou a horizontal table. Into its upper face is let perpendicularly a straight tube which rises to a height of 2 metres above this face, the internal cross-section of the tube being 1 sq. cm. The vessel and the tube are filled with water. Find the pressure on (1) the bottom of vessel, (2) a side, (3) an end, (4) the upper surface, (5) the table.

39. A cylindrical vessel, radius of base 8 in. and height 12 in., is placed vertically on a horizontal table. Into its upper end is let perpendicularly a tube 20 in. in length, the horizontal cross-section of which is 1 sq. in. Compare the pressure on the base and on the curved surface, when the cylinder and the tube are filled with any liquid.

40. A cubical vessel supposed weightless, whose edge is 4 in. in length, is placed on a horizontal table. Into its upper surface is let perpendicularly a straight tube, the horizontal cross-section of which is 1 sq. in. Vessel and tube are filled with a liquid. Find the length of the tube (1) that the pressure on the side of the cube may be four times the pressure on the base before the tube was inserted, (2) that the pressure on the table may be twice the pressure on the base before the tube was inserted.

41. A rectangle, length 6 in., breadth 4, remains immersed vertically in a liquid with one of its long sides in the surface of the liquid. Divide it by a horizontal line into two parts upon which the pressures are equal.

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3. Equilibrium of Liquids of Unequal Density in a Bent Tube.

If two liquids of unequal density which do not mix are poured into a bent tube, they rest in equilibrium in the position shown in Fig. 92, where A and B represent their free surfaces and C their common surface. Let ρ_1 and ρ_2 be the densities of the liquids, and ac and bc the heights of their free surfaces above the horizontal plane CD drawn through C their common surface.

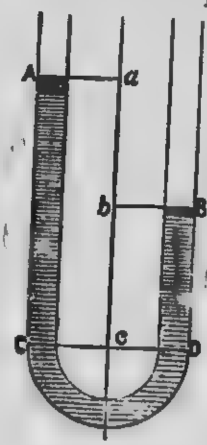

Fig. 92.

Since the liquids are in equilibrium.

The pressure at C = the pressure at D.

But the pressure at $C = \rho_1 \times ac$ × area

and the pressure at $D = \rho_2 \times bc$ × area

$$\therefore \quad \rho_1 \times ac = \rho_2 \times bc$$

$$\text{or} \quad \frac{\rho_1}{\rho_2} = \frac{bc}{ac},$$

Hence,

When the liquids are in equilibrium, their densities are inversely as the heights of their respective columns above their common surface.

EXERCISE XXXIV.

1. Two liquids which do not mix are contained in a bent tube. If their specific gravities are 1.2 and 1.8 respectively, and the height of the first above their common surface is 15 inches, find the height of the other.

2. In a bent tube a column of mercury (sp. gr. 13.6) is balanced by a column of alcohol (sp. gr. .8). If the height of the former is (1) 4 cm., (2) 10 cm., (3) 15 cm., what in each case is the height of the latter?

3. Two tanks are connected by a pipe. Into one tank is poured salt water (sp. gr. 1.03), and into the other petroleum oil (sp. gr. .5). The oil is found to be 5 ft. above their common surface. Find the height of the water.

4. Mercury and ether are poured into a bent tube. The mercury stands 5.25 cm. when ether stands 100 cm. above their common surface. If the density of the ether is .715 grams per cubic centimetre, what is the density of the mercury?

5. Two liquids that do not mix are contained in a bent tube. The difference of their levels is 40 cm. and the height of the denser above their common surface is 70 cm. Compare their densities.

6. If water and a denser liquid which does not mix with it are placed in a U-tube, the internal cross-section of which is 1 sq. cm., the difference of their levels is found to be 5 cm., and the height of the liquid above their common surface is 10 cm. What is the height of the water?

7. The lower ends of two vertical tubes whose cross-sections are 1 and .1 inches respectively, are connected by a tube. The tubes contain mercury (sp. gr. 13.6). What volume of water must be poured into the larger tube to raise the level of the mercury in the smaller tube by one inch?

8. A U-tube has mercury (sp. gr. 13.6) poured into it until the surface of the mercury is half-way up each tube. Water is poured into one branch until it is filled. How high will the mercury rise in the other branch?

9. A U-tube, having its two arms equal, has mercury poured into it until each surface is 6 inches below the top of the tube.

Water is poured into one branch and alcohol (sp. gr. .8) into the other to fill the tube. Find the length of the tube occupied by each liquid.

10. Water is poured into a U-tube, the branches of which are 6 inches long, until they are half full. One of the branches is then filled with oil (sp. gr. $\frac{4}{5}$). What length of the tube does it occupy?

11. A uniform bent tube consists of two vertical branches and of a horizontal portion uniting the lower ends of the vertical portions. Enough water is poured in to occupy 6 cm. of the tube and then enough oil (sp. gr. $\frac{4}{5}$) to occupy 5 cm. is poured in at the other end. If the length of the horizontal part of the tube is 2 cm., find where the common surface is situated.

CHAPTER XVI.

BUOYANCY.

1. Nature of Buoyancy.

When a body is immersed in a liquid, every point of its surface will be subjected to a pressure which is perpendicular to the surface at that point, and which varies as the depth of that point below the surface of the liquid. When these pressures are resolved into horizontal and vertical components, the horizontal components equilibriate each other, and since the pressure on the lower part of the body is greater than that on the upper part, the resultant of all the forces acting upon the body must be vertical and act upward. This force is termed the **resultant vertical pressure** or **buoyancy of the fluid.**

2. What is the amount of the buoyant force which a liquid exerts on an immersed body? Law of Buoyancy.

Experiment.

To answer this question, take a brass cylinder A, which fits exactly into a hollow socket B. Hook the cylinder to the bottom of the socket and counterpoise them on a balance. Surround the cylinder with water (Fig. 93).

It will be found that the cylinder is buoyed up by the water, but that equilibrium is restored when the socket is filled with water. Hence the buoyant force of the water on the cylinder equals the weight of a volume of water equal to the volume of the cylinder.

In general terms, **the buoyant force exerted by a fluid upon a body immersed in it is equal to the weight of the fluid displaced by the body; or a body when**

weighed in a fluid loses in apparent weight an amount equal to the weight of the fluid which it displaces.

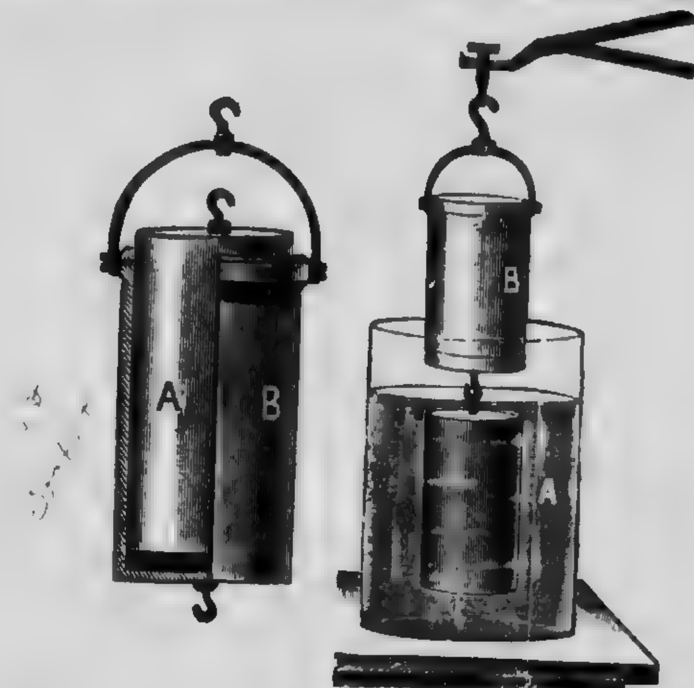

Fig. 93.

This is known as the **principle of Archimedes.**

This conclusion may be arrived at in another way.

Let A be a body which is either wholly immersed (Fig. 94*b*) or which has the part *abcd* (Fig. 94*a*) immersed

Fig. 94*a*.

in a fluid, and suppose the body removed, and its place *abcd* filled up with fluid. Since the fluid body in the space

abcd is of the same density as the surrounding fluid, it will remain in equilibrium. The forces acting upon it are:

FIG. 94b.

(i) Its weight.

(ii) The resultant fluid pressure upon its surface.

Therefore the resultant fluid pressure equals the weight of the fluid *abcd*. Since the weight of the fluid acts vertically downward through its centre of gravity, the resultant pressure must act vertically upward through the same point. If now we suppose the solid body A to replace the fluid in the space *abcd*, it is evident that the resultant pressure of the fluid will thrust it upward with the same force, that is, with a force equal to the weight of the fluid displaced by the solid.

EXERCISE XXXV.

1. A cu. ft. of marble which weighs 300 pounds is immersed in water. Find (1) the resultant fluid pressure upon it, (2) its weight in water.

2. A cu. in. of a metal weighs 5 pounds in air. Find its weight in water.

3. Three and one-half c.dcm. of a substance weigh 6 kgm. Find the weight when immersed in water.

4. Six and three-fourths c.cm. of a substance weigh 18.5 grams. Find the weight when immersed in a liquid twice as heavy as water.

5. A body whose volume is $2\frac{3}{4}$ cu. ft. weighs 420 pounds. Find its weight when $\frac{2}{3}$ of its volume is immersed in water.

6. A substance whose volume is $3\frac{1}{4}$ c.dcm. weighs $7\frac{5}{8}$ kgm. Find its weight when $\frac{2}{3}$ of its volume is immersed in a liquid one-half as heavy as water.

7. Five cu. metres of a metal weigh *in vacuo* 20,000 kgm. With what force would it be buoyed up if it were suspended in air? Find its weight in air. 1 c.cm. of air weighs .0013 grams.

8. Three and one-half cu. ft. of granite weigh 750 pounds in water. Find its weight.

9. A body whose volume is $4\frac{1}{3}$ c.dcm. weighs in water $5\frac{1}{4}$ kgm. Find its weight.

10. A substance whose volume is m cu. feet weighs n pounds in air. Find its real weight, that is its weight *in vacuo*, if a cu. inch of air weighs p oz.

11. Seven and one-half c. metres of a substance weigh 9,000 kgm. when $\frac{2}{3}$ of its volume is immersed in water. Find its weight.

12. A body, the volume of which is $5\frac{3}{8}$ cu. feet, weighs 75 pounds when $\frac{3}{11}$ of its volume is immersed in a liquid $\frac{2}{3}$ as heavy as water. Find its weight.

13. Find the volume of a body that weighs 10 kgm. in air and 8 kgm. in water.

14. Find the volume of a cube of iron which weighs 2,800 pounds in air and 2,425 pounds in water.

15. Find the edge of a cube of lead which weighs 90.8 kgm. in air and 84.8 kgm. when $\frac{3}{4}$ of its volume is immersed in water.

16. A cu. in. of one of two liquids weighs $\frac{3}{4}$ oz., and of the other $\frac{1}{3}$ oz. A body immersed in the first weighs 7 oz., and in the second 12 oz. Find the weight and the volume of the body.

17. A c.cm. of water weighs 1 gram and a c.cm. of air weighs 0.0013 grams. A body weighs 100 grams in air and 40.078 grams in water. Find (1) its weight *in vacuo*, (2) its volume.

18. The mass of a piece of limestone (sp. gr. = 2.637) is 256.34 gm. What is its apparent weight in water?

19. The apparent weight of a mineral when weighed in water is 195.46 gm. If its specific gravity is 2.678, what is its mass?

20. Find the apparent weight of 5 c.cm. of gold (sp. gr. = 19.3) in mercury (sp. gr. = 13.6).

21. A vessel of water is placed in one scale pan of a balance and counterpoised. Will the equilibrium be disturbed if a person dips his finger into the water without touching the sides of the vessel? Explain fully.

22. A vessel containing water is placed in one scale pan of a balance, and balanced by weights in the other scale pan. A piece of lead (sp. gr. = 11.4) weighing 17.1 gm. is suspended by a string from a fixed support, and is placed totally immersed in the water. What additional weight must be used to restore the equilibrium? In which scale pan must it be placed?

23. Water floats upon mercury, whose specific gravity is 13, and a mass of platinum, whose specific gravity is 21, is held suspended by a string so that $\frac{19}{24}$ of its volume is immersed in the mercury and the remainder of its volume in the water. Prove that the tension of the string is half the weight of the platinum.

24. Eight cubic centimetres of a metal whose specific gravity is 6, and a certain volume of platinum whose specific gravity is 21 are connected by a fine thread passing over a smooth pulley, and rest in equilibrium when both bodies are immersed in water. By how much must the metal be diminished in volume to preserve equilibrium (1) when the metal is removed from the water, (2) when both the metal and the platinum are removed from the water?

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3. Conditions of Equilibrium of a Body acted upon by Fluid Pressure.

When a body is placed in a fluid and left to itself two forces act upon it.

(i) The force of gravity, acting vertically downward through the centre of gravity of the body.

(ii) The resultant fluid pressure, acting vertically upward through the centre of buoyancy (the centre of gravity of the fluid displaced) = the weight of the fluid displaced.

If the body is in equilibrium under the action of these forces, they must be equal and act in opposite directions in the same straight line. The following are, therefore, the conditions of equilibrium when a body floats wholly or partially immersed in a fluid.

1. The weight of the body equals the weight of the fluid displaced by it.

2. The centre of gravity of the body is in the same vertical line as the centre of buoyancy.

EXERCISE XXXVI.

1. A cubic in. of pine floats with $\frac{5}{7}$ of its volume in water. Find its weight.

2. A c.cm. of poplar floats with $\frac{m}{n}$ of its volume out of water. Find its weight.

3. The weight of $2\frac{1}{4}$ cu. feet of elm is 124 pounds. What part of its volume will be immersed if it is allowed to float in water?

4. The weight of $6\frac{2}{3}$ c.dcm. of cork is $1\frac{2}{3}$ kgm. If it is allowed to float in water, how many c.dcm. will remain above the surface?

5. A piece of wood weighing 100 pounds floats in water with $\frac{2}{3}$ of its volume above the surface. Find its volume.

6. If a piece of ash (sp. gr. $=.8$) is allowed to float in water, what part of its volume will be immersed?

7. A cylinder 12 in. long made of larch (sp. gr. $=.53$) floats in water. How many inches will remain out of water?

8. If a body whose specific gravity is 4 float in a liquid whose specific gravity is 5, what portion of the body will be immersed?

9. Seven and one-half cu. ft. of poplar floats with $\frac{2}{3}$ of its volume out of a liquid (sp. gr. $=.8$). Find its weight.

10. A piece of pine weighs $a\,n$ grams and floats with $\frac{a}{n}$ of its volume in water. Find its volume.

11. What is the least force which must be applied to a cu. ft. of larch which weighs 30 pounds that it may be wholly immersed in water?

12. A c.dcm. of cork, weighing 480 grams, floats just immersed in water, when prevented from rising by a string attached to the bottom of the vessel containing the water. Find the tension of the string.

13. A cylindrical cup weighs 35 grams, its external radius being $1\frac{3}{4}$ cm., and its height 8 cm. If it be allowed to float in water with its axis vertical, what additional weight must be placed in it that it may sink?

14. A cylinder of wood, 8 in. long and weighing 15 pounds, floats vertically in water with 3 in. of its length above the surface. What is the tension of the string which will hold it just immersed in water?

15. A cylindrical buoy, 10 ft. long and horizontal cross-section 4 sq. ft., weighs 860 pounds. If it be anchored with one-half of its volume in sea-water, find the tension of the anchor chain (1 cu. ft. of sea-water weighs $64\frac{1}{2}$ pounds).

16. A cube floats in water with its upper face in the surface when a weight of $62\frac{1}{2}$ pounds is placed upon it. If the cube rises 1 in. when the weight is removed, find the edge of the cube.

17. A cylinder floats vertically in water, and sinks 2 in. when a weight of 250 oz. is placed upon it. Find the horizontal cross-section of the cylinder.

18. A body floats with one-third of its volume immersed in a liquid. If it requires a weight of 10 pounds to cause it to float with one-half of its volume immersed, find its mass.

19. The area of the cross-section of a ship at the water-line is 10,000 sq. ft. What additional load will sink it $2\frac{2}{5}$ in.?

20. A cylinder floats vertically in a liquid. Compare the forces necessary to raise it 3 in. and to depress it 3 in.

21. What is the least weight that must be placed upon a cu. ft. of cork (sp. gr. = .25) that it may float totally immersed in a liquid whose specific gravity is .9?

20

22. What is the least weight that must be placed upon a piece of wood weighing 20 pounds and floating with $\frac{2}{3}$ of its volume immersed in a liquid whose specific gravity is 1.5 that it may be totally immersed?

23. A cylinder of cork weighs 10 grams, and its specific gravity is .25. Find the least force that will immerse it (1) in water, (2) in a liquid whose specific gravity is .75.

24. A body (sp. gr. = .5) floats on water. If the weight of the body is 1 kgm., find the number of cubic centimetres of it above the surface of the water.

25. If a cube floats on water with one of its faces horizontal, and a body the mass of which is 9 grams when placed upon it makes it sink 1 cm., find the size of the tube.

26. A cylinder of wood floats in water with three-fourths of its volume above the surface; when a cylinder of metal half as large again is attached to the first, the two float just immersed. Compare the densities of the wood and the metal.

27. A hollow cubical metal box of which the length of an edge is one inch and the thickness one-eighteenth of an inch, will just float in water when a piece of cork of which the volume is 4.34 c. in. and the specific gravity .5, is attached to the bottom of it. Find the density of the metal.

28. A cylinder of larch 19 cm. in height is joined to a cylinder of iron (sp. gr. = 7.8) 1 cm. in height so as to form one cylinder 20 cm. in height. This is found to float in water with two cm. projecting above the surface. Find the density of larch.

29. A rod of uniform section is formed partly of platinum (sp. gr. = 21) and partly of iron (sp. gr. = 7.5) the platinum portion being 2 in. long. What will be the length of the iron portion when the whole floats in mercury (sp. gr. = 13.5) with the iron 1 in. above the surface?

30. A liquid (sp. gr. = 1.6) is poured into a vessel containing mercury (sp. gr. = 13.1), and a cylinder of zinc (sp. gr. = 7) allowed to sink through the liquid floats with its axis vertical in the mercury. If the cylinder is 5 dcm. long, find the length of the part immersed in the mercury.

10

31. Taking 7.67 pounds as the weight of 100 cu. ft. of air, find approximately the volume of hydrogen (sp. gr. compared with air .07) which a balloon must contain in order that its lifting power may be equal to a weight of 713 lbs.

32. A body floats in a fluid (sp. gr. = .9) with as much of its volume out of the fluid as would be immersed if it floated in a fluid (sp. gr. 1.2). Find the specific gravity of the body.

33. A cubical block of wood (sp. gr. = .6) whose edge is 1 foot floats, with two faces horizontal, down a fresh water river out to sea, where a fall of snow takes place, causing the block to sink to the same depth as the river. If the specific gravity of the sea water is 1.025, find the weight of the snow on the block.

34. A ship, of mass 1,000 tons, goes from fresh water to salt water. If the area of the section of the ship at the water-line is 15,000 sq. ft., and her sides vertical where they cut the water, find how much she will rise, taking the specific gravity of sea water as 1.026.

35. A piece of iron (sp. gr. = 7.5) the mass of which is 26 lbs., is placed on the top of a cubical block of wood floating in water, and sinks it so that the upper surface of the wood is level with the surface of the water. The iron is then removed. Find the mass of the iron that must be attached to the lower surface of the wood so that the top may be as before in the surface of the water?

36. A body floats in water contained in a vessel placed under an exhausted receiver with one-half its volume immersed. Air is then forced into the receiver until its density is 80 times that of the air at the atmosphere pressure. Prove that the volume immersed in the water will then be $\frac{4}{9}$ of the whole volume assuming the specific gravity of air at the atmospheric pressure to be .00125.

37. An accurate balance is completely immersed in a vessel of water. In one scale pan some glass (sp. gr. = 2.5) is being weighed and is balanced by a one-pound weight whose sp. gr. is 8, which is placed in the other scale pan. Find the real weight of the glass.

38. A cubical box, the volume of which is 1 cu. ft., is three-fourths filled with water, and a leaden ball, the volume of which is 72 cu. in.,

is lowered into the water by a string. Find the increase in pressure on (1) the base, (2) a side of the box.

39. A cylindrical bucket, 10 in. in diameter and one foot high, is half filled with water. A half-hundredweight of iron is suspended by a thin string and held so as to be completely immersed in water without touching the bottom of the bucket. Subsequently the string is removed and the iron allowed to rest on the bottom of the bucket. By how much will the pressure on the bottom be increased in each case by the presence of the iron? (A cu. ft. of iron weighs 440 lbs.)

CHAPTER XVII.

ATMOSPHERIC PRESSURE.

1. Has air weight?

Experiment 1.

Take a vessel A (Fig. 95), which can be attached to an air-pump, weigh it, exhaust the air from it, close the stop-cock, and weigh it again.

1. What difference in weight is observed?

2. What causes this difference?

Allow the air to re-enter and observe the result.

1. Has air weight?

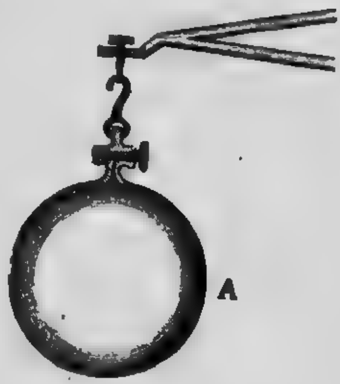

FIG. 95.

Gases, like solids and liquids, possess Weight.

2. Pressure due to weight.

We have seen that, on account of their weight, solids exert pressure on the bodies which support them, and liquids exert pressure on all bodies in contact with them. **Do gases exert pressure?**

Experiment 2.

Tie a piece of sheet rubber over the mouth of a thistle-tube and exhaust the air from the tube by suction, or by

connecting it by means of a piece of heavy rubber tubing with an air-pump or an aspirator (Fig. 96).

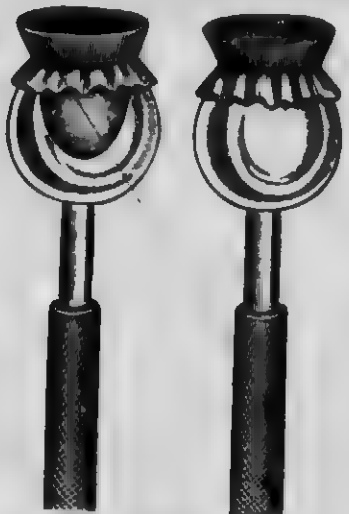

FIG. 96.

1. What change takes place in the position of the rubber membrane?

2. What causes this change?

Turn the tube so that the membrane may face upwards, downwards, and in various directions.

1. Does the position of the membrane change as the tube is turned in different directions?

2. What does this prove with regard to the intensity of the pressure of the air in different directions at the same point?

Experiment 3.

Place a receiver on the plate of an air-pump, and exhaust the air from the receiver (Fig. 97).

Try to separate the receiver from the plate.

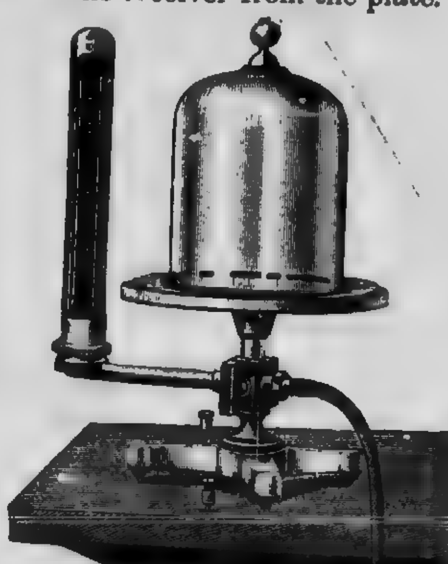

FIG. 97.

1. What evidence have you that the air on the outside of the receiver presses downward upon it?

2. What evidence have you that the air which was within the receiver exerted an upward pressure on it?

Gases, like liquids, on account of their weight, exert pressure on the surfaces of bodies immersed in them, and this pressure is equal in all directions at the same point.

3. Pressure Due to the Expansive Force of a Gas.

We have seen (Art. 2, page 170) that gases tend to expand indefinitely, and that they consequently exert pressure on the surfaces of the vessels that contain them. This action may be illustrated by additional experiments.

Experiment 4.

Fill a bottle partly full of water, cork it with a perforated cork, and connect it by a bent tube with an uncorked bottle, as shown in Fig. 98. Place both bottles under the receiver of an air-pump, and exhaust the air from the receiver.

1. What movement takes place in the water?

2. What must have caused it?

3. Why did not this force cause the movement in the water before the air was exhausted from the receiver?

Let the air into the receiver again.

What takes place? Why?

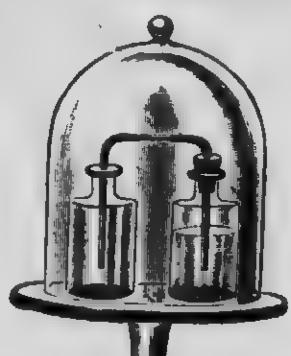

FIG. 98.

Experiment 5.

Take a long glass tube A, closed at one end, and fitted at the other with a stop-cock which screws into the plate of an air-

pump. (The tube known as the Guinea-and-Feather Tube answers well.) Stand the tube in a vertical position, with the open end of the tap in water (Fig. 99). Open the tap.

FIG. 99.

Does the water rise in the tube?

Take the tube out of the water, screw it to the air pump and partially exhaust the air, close the tap, unscrew it from the pump and place it as before in water. Open the tap.

1. What is the cause of the movement in the water?

2. Did the pressure which caused this movement exist before the air was removed from the tube? If so, why did not the movement take place?

3. Is this pressure equal to the expansive force of the air within the tube when the water comes to rest? Give reasons for your answer.

4. When the tube was placed in the water and the tap opened, what change took place in (*a*) the density of the air remaining in the tube, (*b*) its mass, (*c*) its expansive force.

Gases, on account of their tendency to expand indefinitely, exert an expansive force, which is of equal intensity at all points both within the mass of the gas itself and upon the internal surface of the vessel which contains it.

4. Measure of the Rate of Pressure of the Atmosphere.—The Barometer.

The pressure of the atmosphere may be measured as other forces often are, by measuring some counter-balancing force.

Experiment 6.

Connect a glass tube A closed at one end with another B of the same size, but open at both ends, by a piece of stout rubber tubing C (Fig. 100). Each glass tube should be about 80 cm., and the rubber tube about 15 cm. in length and 4 mm. in diameter. Hold the tubes in the position shown at the left hand, and fill A and the rubber tube with mercury. Now invert A and place the connected tubes in the position shown at the right hand, thus forming a U-shaped tube, of which the branches are A and B.

1. What is the length of the column of mercury in A above the level of the mercury in B?

The weight of this column of mercury is just balanced by the weight of the column of air pressing on the surface of the mercury in B. Hence the pressure of the air on the surface of the mercury in B may be measured by the weight of the mercury in A above the level of the mercury in B.

FIG. 100.

1. What is the length of the column of air which weighs the same as the column of mercury in A above the level of the mercury in B?

2. What transmits the air pressure on the surface of the mercury in B to column of mercury sustained by it?

3. Devise an experiment to show that the column of mercury in A is sustained by the pressure of the air on the surface of the mercury in B.

4. If the tube A and B were of different diameters, would the difference in levels of the mercury in the two tubes be the same as in this case, where the tubes are of equal diameter? Why?

A tube of this form permanently mounted and supported with scales for determining the differences in level between the mercury in the two branches is one form of the **barometer**, an instrument used for determining the pressure of the atmosphere. Fig. 101 shows a barometer of this form.

Fig. 101

The upper scale gives the height of the mercury in the closed branch above a fixed point, and the lower scale the distance of the mercury in the open branch below the same fixed point. The sum of the two readings is the height of the barometer column.

Instead of a U-shaped tube, a straight tube is frequently used to contain the mercury in measuring the pressure of the atmosphere.

Fig. 102.

Experiment 7.

Take a glass tube, about one centimetre in diameter and 80 centimetres long, closed at one end, fill it with mercury and, stopping the open end with the finger, invert it and place it in a vertical position with the open end under the surface of the mercury in another vessel (Fig. 102).

1. What takes place when the finger is removed? Explain the reason. See Experiment 5, page 211.

When a straight glass tube is used as a barometer, the cistern which contains the mercury has usually a flexible leather bottom which can be moved up or down by a screw C (Fig. 103). A scale is attached to the side of the tube by means of which the height of the surface of the mercury in the tube above a fixed point in the cistern may be observed.

To read the barometer the screw C is turned until the surface of the mercury in the cistern comes to the fixed point. The scale will then indicate the difference in the mercury levels in the tube and the cistern.

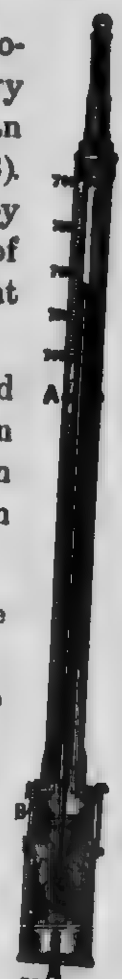

FIG. 103.

1. Upon what does the air press which sustains the column of mercury?

2. Would the height of this column be changed if the tube were not of uniform bore? Why?

3. What change in the height of the column would indicate an increase in the pressure of the atmosphere? What change a decrease? Why?

4. What is there in the tube above the mercury?

5. What effect would be produced by admitting a little air into this space? What force produces this effect?

6. Which would be the more suitable for an accurate barometer, a tube of fine bore or one of wide bore? Explain.

7. Explain why a barometer falls when carried up a mountain.

5. Find the Pressure of the Atmosphere on a Surface the Area of which is a, when the Height of a Barometer Using a Liquid of Density ρ is h.

The pressure of the atmosphere on the surface equals the weight of a column of the liquid used in the barometer whose base is of area a and whose height is h.

But the weight of the liquid

$$= \text{its volume} \times \text{its density}$$
$$= (a \times h) \times \rho.$$

Hence the atmospheric pressure on the surface

$$= ah\rho.$$

EXERCISE XXXVII.

Note.—In the following questions the density of mercury is to be taken as 13.6 grams per cubic centimetre, or as 13,600 oz. per cubic foot.

1. Find the atmospheric pressure per sq. inch when the mercury barometer stands at 30 inches.

2. Find the pressure of the atmosphere on a square centimetre when the mercury barometer stands at 76 cm.

3. Three barometers are constructed to use liquids whose specific gravities are respectively 7.2, 2.9, and 11.8. Find the atmospheric pressure on a sq. inch (1) when the first barometer stands at 4.8 ft., (2) when the second stands at 11.52 ft., (3) when the third stands at 5.76 cm.

4. Three barometers are constructed to use liquids whose specific gravities are respectively 13.6, 5.17, and 2.06. Find the atmospheric pressure on 1 sq. cm., (1) when the first barometer stands at 70 cm., (2) when the second stands at 2 metres, (3) when the third stands at 5 metres.

5. If in ascending a mountain the barometer falls from 30 in. to 20 in., find the decrease in the atmospheric pressure on an area of 10 sq. ft.

6. If the atmospheric pressure is 15 pounds per sq. in., find the heights of the columns in barometers constructed to use liquids whose specific gravities are respectively 1.44, 3.6, and 4.8.

7. The atmospheric pressure is 1,033.6 grams per sq. centimetre. Find the heights of the columns in barometers constructed to use liquids whose specific gravities are respectively 13.6, 6.8 and 3.4.

8. Find the heights of the columns in barometers constructed to use liquids whose specific gravities are respectively 1.5, 1.7, and 5 when the mercury barometer stands at 30 in.

9. The mercury barometer stands at 76 cm. What is the height of a water barometer?

10. A mercury barometer stands at 28.8 in. Find the sp. gr. of nitric acid, if a column of it 22.72 ft. in height can be supported by the atmospheric pressure.

11. Glycerine rises in a barometer tube to a height of 26 ft. when mercury stands at 30 in. What is the density of glycerine?

12. At the surface of a lake the barometer stands at 30 in. What will be the reading of the barometer when it is sunk in the water to a depth of 100 ft.?

13. At what depth below the surface of a lake will the barometer stand at 80 in., if at the surface it stands at 30 in.?

14. A mercurial barometer is sunk in a vessel of water, and reads 32.18 in. when its free surface is 3 ft. below the surface of the water, while outside the reading is 29.8 in. Compare the density of mercury with that of water.

15. Solve the following questions in Exercise XXXIII, taking the atmospheric pressure into account:—(*a*) Nos. 1, 8, 13, 18, if the barometer stands at 30 in. (*b*) Nos. 2 and 28, if the barometer stands at 76 cm.

16. Into the upper surface of a closed vessel is let perpendicularly a straight tube. The vessel is completely filled with water, and the tube is also filled to a height of 11 ft. 4 in. Find the upward

pressure of the water on 1 sq. ft. in the upper surface of the vessel, if the height of the water barometer is 34 ft.

17. Water floats on mercury to a depth of 20 ft.; at what depth below the surface of the mercury will the pressure on the area of 1 sq. ft. be 20,266 pounds, when the barometer stands at 30 in.?

---

6. The Relation between the Volume and the Pressure of a Gas—Mariotte's Law.

Experiment 8.

Take a tube about 25 cm. long and at least 4 mm. in diameter, one end of which is closed by a stop-cock. A thistle-tube supplied with a stop-cock answers well. Connect this by means of a heavy rubber tube not less than 50 cm. long with a glass tube, also about 50 cm. long. The joints should be wrapped with fine wire or string. Place the tube in a support as shown in Fig. 104, open the stop-cock and pour mercury into the connected tubes until it reaches the same level at or near the centre of each glass tube. Close the stop-cock. Take the reading of the barometer.

Height of barometer (H)= ?

The pressure to which the enclosed air is subjected is measured by,

(1) barometric reading (H)
when the mercury surfaces are at the same level. Why?

(2) The barometric reading (H)± the difference between the levels of the mercury surfaces
when these surfaces are not at the same level. Why?

The plus sign is to be taken when the mercury in the open tube is higher, and the minus sign when it is lower than in the closed tube. Why?

Place the open tube in several positions, with the surface of the mercury in it either above (Fig. 105) or below (Fig. 106) the surface of the mercury in the closed tube; and measure

(1) The lengths of the air column in the closed tube;

(2) The vertical distances between the mercury levels in the two tubes.

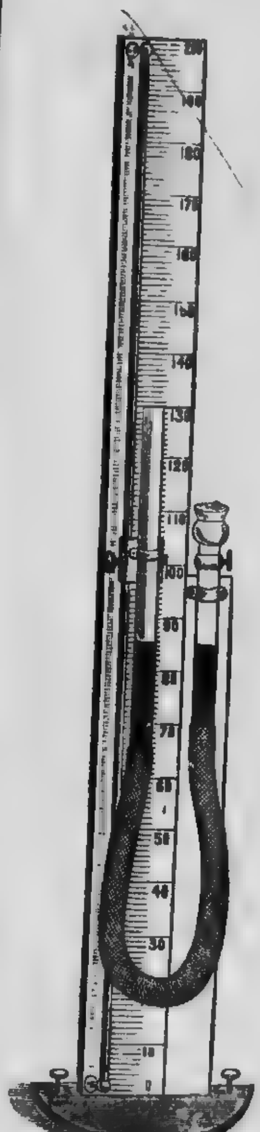

FIG. 104.

FIG. 105.

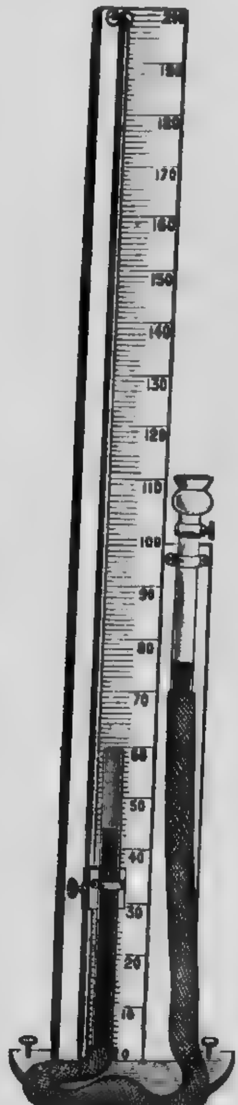

FIG. 106

Supposing that V represents the original volume of the enclosed air, and H the reading of the barometer; and that

V_1, V_2, V_3, V_4, etc., represent the volumes of this air at successive observations; and that H_1, H_2, H_3, H_4 represent the differences in mercury levels for these observations, fill up the following table :—

VOLUMES.	PRESSURES.	PRODUCTS.
$V =$	$P = H =$	$V \times P =$
$V_1 =$	$P_1 = H \pm H_1 =$	$V_1 \times P_1 =$
$V_2 =$	$P_2 = H \pm H_2 =$	$V_2 \times P_2 =$
$V_3 =$	$P_3 = H \pm H_3 =$	$V_3 \times P_3 =$
$V_4 =$	$P_4 = H \pm H_4 =$	$V_4 \times P_4 =$
Etc.	Etc.	Etc.

If the experiment is carefully performed, the products $V \times P$, $V_1 \times P_1$, etc., will be found to be equal. This being the case, it is evident that the volume of the air is decreased at exactly the same rate as the pressure is increased, or is increased at the same rate as the pressure is decreased. That is, the volume of a given portion of air varies inversely as the pressure to which it is subjected.

The extended researches of careful experimenters have shown that all gases, within certain limitations, conform to this law.

The law is known as Boyle's or Mariotte's Law. It may be thus stated :—

7. Boyle's or Mariotte's Law.

If the temperature is kept constant, the volume of a given mass of gas varies inversely as the rate of pressure to which it is subjected.

The gases which most closely follow this law are those which are farthest removed, both as to temperature and pressure, from their points of liquefaction.

When a gas nears its liquefying point, the reduction in volume is greater than that which the law would indicate.

EXERCISE XXXVIII.

1. If the volume of the air shut up in the tube, Experiment 8, page 218, is 10 c.cm. when the mercury is at the same level in each tube and the barometer stands at 70 cm., what will be the difference in level between the surfaces of the mercury in the tubes when the volume of this air occupies (*a*) 5 c.cm. (*b*) 20 c.cm.?

2. The differences in levels, Experiment 8, page 218, at four different observations were 10 cm., 90 cm., 170 cm., 250 cm., and the volume of the enclosed air at the first observation was 12 c.cm. What was the volume of the air at each of the other observations if the barometer stands at 70 cm.?

3. What effect would (*a*) raising, (*b*) lowering, the open tube, Experiment 8, page 218, have upon (1) the mass, (2) the density, (3) the expansive force of the enclosed air?

4. The pressure of a gas is 10 grams per sq. cm. when its volume is 100 c.cm., what is the pressure when the volume is 150 c.cm.?

5. The volume of gas shut up in a rubber bag is 200 c.cm. when the barometer stands at 70 cm. What will be the volume of the gas when the barometer stands at 80 cm.?

6. If a gas occupies a volume of 25 c.cm when the barometer stands at 76 cm., what must be the reading of the barometer when the gas measures 30 c.cm.?

7. A gas holder contains 22.4 litres of a gas measured when the barometer stands at 72 cm. What will be the volume of the gas when the barometer stands at 76 cm.?

8. A rubber bag contains 100 c.cm. of air at the atmospheric pressure, what will the volume of the air become if the bag is sunk to a depth of 30 feet in water? What would be the buoyant force of the water upon it? The water barometer stands at 30 feet.

9. Why is compressed air used in (*a*) bicycle tires, (*b*) air-cushions?

8. Air-Pump.

The air-pump is used for removing the air from enclosed vessels.

Fig. 107 shows one of the most common forms of the air-pump. A cylindrical barrel AB is connected by means of a pipe C with a receiver R, from which the air is to be removed. A piston D, in which there is a valve opening upward, is worked in the barrel by a rod which passes through the air-tight collar at the top of the barrel. At A and B, the ends of the barrel, are valves opening upward. A gauge G for testing the extent of the exhaustion is sometimes connected with the tube C by means of a tap T.

Suppose D at its lowest position. As it ascends, the compression of the air in AD closes the valve in the piston (Fig. 107) and opens the valve A, and the enclosed

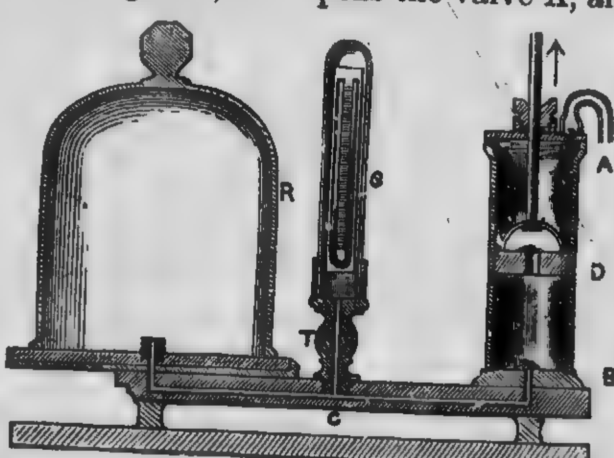

Fig. 107.

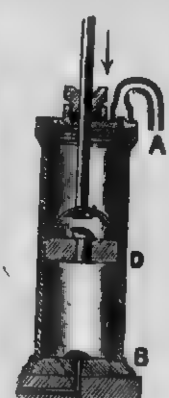

Fig. 108.

air escapes, while a part of the air in the receiver R flows through the valve B and occupies the vacuum formed below D. When the piston begins to descend, the valves A and B are closed (Fig. 108), and the air in DB flows

up through the valve in D. Thus at each double stroke of the piston a fraction of the air is removed from the receiver.

1. Of what use is the valve A?

2. What causes it to close when the piston descends?

3. What causes the valve in D to open and the valve B to shut when the piston descends?

9. Condenser.

It is frequently necessary to pump air into a receiver, as in filling the tubes of a bicycle. In this case the valves of the pump open downwards as shown in Fig. 109 instead of upwards as in the pump for withdrawing the air from the receiver.

Fig. 109.

1. Explain the action of the valves B and D on (*a*) the down-stroke, (*b*) the up-stroke, and show how the density of the air in the vessel C is increased by the working of the pump.

2. Obtain a small bicycle pump, take it apart and study its construction and action.

10. To Determine the Pressure and the Density of the Air in the Receiver of an Air-Pump, after *n* Strokes of the Piston.

Let V and v denote the volumes of the receiver and the barrel respectively, P the pressure of the air at first, $P_1, P_2, P_3 \ldots P_n$ the pressures of the air after 1, 2, 3 . . . n strokes of the piston.

When the piston is first raised, the air in the receiver expands and occupies both the receiver and the barrel.

Thus a volume V of air becomes $V + v$.

But by Mariotte's Law the pressure of the air varies inversely as volume. (Page 220.)

Therefore $$\frac{P}{P_1} = \frac{V + v}{V}$$

or $$P_1 = \frac{V}{V + v}P$$

Similarly $$P_2 = \frac{V}{V + v}P_1$$

$$= \left(\frac{V}{V + v}\right)\left(\frac{V}{V + v}\right)P = \left(\frac{V}{V + v}\right)^2 P$$

And so on

Hence finally $$P_n = \left(\frac{V}{V + v}\right)^n P.$$

The density of the air varies directly as the pressure, hence if ρ denotes the density of the air at first, and $\rho_1\ \rho_2\ \rho_3 \ldots \rho_n$, the densities after 1, 2, 3 . . . . n strokes

$$\rho_n = \left(\frac{V}{V + v}\right)^n \rho.$$

Why cannot all the air be removed from the receiver by means of the air-pump?

EXERCISE XXXIX.

1. The capacity of the receiver of an air-pump is five times that of the barrel. Compare the density of the air after the fifth stroke with the density at first.

2. If the capacity of the receiver of an air-pump is four times that of the barrel, and the initial density of the air is 1, find the density after (1) the 3rd stroke, (2) 5th stroke, (3) 8th stroke of the piston.

3. The capacity of the receiver of an air-pump is seven times that of the barrel. After how many strokes will the density of the air in the receiver be to the initial density as 343 : 512?

4. The volume of the barrel of an air-pump is 24 cu. in., when the volume of the receiver is 72 cu. in. The expansive force of the air

in the receiver supports a column of mercury 30 inches in height. Find the height of the column of mercury supported after (1) the 5th, (2) the 15th, (3) the 20th stroke.

5. The volume of the receiver of an air-pump is ten times that of the barrel, and a barometer placed under the receiver stands at first at 732.05 mm.; but after a certain number of strokes it stands at 500 mm. Find the number of strokes.

6. Find the ratio of the volume of the receiver to that of the barrel of an air-pump, if at the end of the 3rd stroke the density of the air in the receiver is to the original density as 1331 : 1728.

7. A barometer under the receiver of an air-pump stands at 768 mm., but after four strokes of the piston it stands at 243 mm. Find the ratio of the volume of the receiver to that of the barrel.

8. If the receiver of an air-pump holds 100 gm. of air at the ordinary pressure and the barrel holds 10 gm., what will be the weight of the air in the receiver after the third stroke?

9. Show how to find the density and the pressure of the air within a condenser after n strokes.

10. The volume of the receiver of a condenser is twelve times that of the barrel. Compare the density of the air after the sixth stroke with its original density.

11. The volume of the barrel of a condenser is 10 cu. in., and that of the pneumatic tire of a bicycle into which the air is forced is 100 cu. in. If there is a weak point in the tire just capable of sustaining a pressure of 15 pounds to the square in., after how many strokes of the piston will the tire burst, the original pressure of the air being 15 pounds to the square inch?

11. Common Pump.

This pump is used for drawing water from a well or cistern. The construction is shown in Fig. 110. A cylindrical barrel AB is joined to one end of a suction-pipe BC, the other end of which is placed in the water to be drawn. A piston D, in which there is a valve opening upward, is worked in the barrel by means of a piston-rod. At B there is a valve opening upward.

At G a hole is made in the barrel and a spout is inserted.

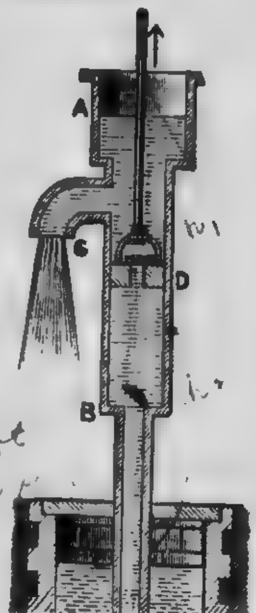

Fig. 110.

34 ft high

Suppose the piston D at its lowest position. As it ascends the valve in D is closed, while the air in the suction-pipe expands, opens the valve B (Fig. 110), and a part of it occupies the vacuum formed below D. Now, since the air which was in the suction-pipe BC occupies a larger space DC, its pressure on the water in the tube will be less than that of the external air upon the surface of the water at C; consequently this external pressure will cause the water to rise until equilibrium is restored, that is, until the expansive force of the air below D, together with the pressure at C due to the weight of the column of water above it, equals the external pressure of the atmosphere at C. (See Exp. 5, page 211.) When the piston descends, the valve B is closed and the valve in D opened by the compression of the air in DB (Fig. 110*a*), and the air in the lower part of the barrel passes through the valve in B, while the water remains at the same level in the suction-pipe.

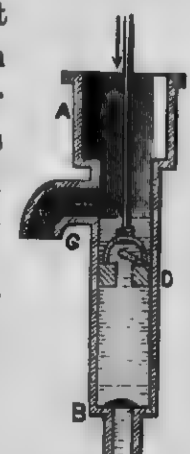

Fig. 110*a*.

At each subsequent stroke the water rises still further in the suction-pipe, and at length forces open the valve B, enters the barrel, passes up through the valve in the piston when it is descending, and is carried forward and thrown out at the spout G when the piston is reascending.

1. What is the greatest length which BC can have? Why?

2. How could a common pump be used to lift water from a very deep well?

3. Why does the water stand in the suction-pipe when the piston is not being worked?

4. A small hole is frequently made in the suction-pipe of a pump to prevent the freezing of the water in the pipe. If such a hole is bored, how high will the water stand in the pipe when the piston is not being worked? Why?

5. The top of a well is covered and sealed air-tight. How will this affect the working of the pump (1) when the well is filled with water, (2) when it is partially filled? Explain.

12. Force Pump.

The construction is shown in Fig. 111. A cylindrical barrel AB is joined to one end of a suction-pipe BC, the other end of which is placed in the water to be drawn. A solid piston is worked in the barrel by a piston-rod. At D a hole is made in the barrel and a pipe E inserted. At B and D are valves opening outward.

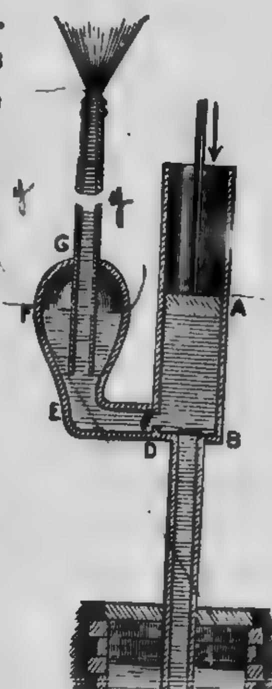

FIG. 111.

Suppose the piston at its lowest position. As it ascends the air in BC expands, opens the valve B and a part of it occupies the vacuum formed below the piston, the water being forced up into the pipe BC by the pressure of the air upon the surface of the water in the well.

When the piston descends, the air in AB being compressed closes the valve B and flows out through the valve D. At each subsequent stroke of the piston the water rises higher in the suction-pipe, and at length flows into the barrel, when by the descent of the piston the valve B is closed and the water forced through the valve D (Fig. 111) into the tube E. If the pipe G is connected directly with the pipe E, the stream flowing through it will be intermittent, as it is only on the descent of the piston that the water is forced through D. To produce a continuous stream an interruption is made in the tube which is surrounded by a strong air-tight vessel F. When the water is pumped into this vessel and rises above the open end of the tube G, which is somewhat smaller than the pipe E, the air in the vessel is compressed, and by its expansive force presses the water through the pipe G even when the piston is ascending, thus producing a continuous stream (Fig. 111a).

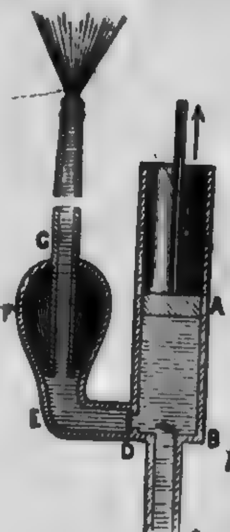

FIG. 111a.

13. Tension of the Piston-rod in a Common Pump and in a Force Pump.

The tension of the piston-rod is the difference between the pressures on the upper and the lower surfaces of the piston.

If H is the measure of the atmospheric pressure, h_1 the height of the water above the piston, and h_2 the height of the column of water between the piston and the surface of water in the well, the pressure on the upper

surface of the piston is $H+h_1$, while that on the lower surface is $H-h_2$, since the weight of the column h_2 counterbalances a part of the atmospheric pressure. Therefore the tension of the piston-rod is

$$(H+h_1)-(H-h_2)=h_1+h_2,$$

Hence,

The force necessary to raise the piston in the common pump is equal to the weight of a column of water whose base is the surface of the piston and whose height is the vertical distance between the levels of the water in the pump and the well.

What is the tension of the piston-rod of the force pump on (1) the up-stroke, (2) the down-stroke?

X Y at

EXERCISE XL.

1. What is the greatest height to which water can be raised by a common pump when the mercury barometer stands at 76 cm., the sp. gr. being 13.6?

2. How high can sulphuric acid be raised by a common pump when the mercury barometer stands at 27 in., the sp. gr. of sulphuric acid being 1.8 and that of mercury being 13.6?

3. The area of a piston of a common pump is 4 sq. in. Find the tension of the piston-rod when the water stands in the suction-pipe at a height of 12 feet above the water in the well.

4. If the spout of a common pump is 21 feet above the surface of the water in a well, and the diameter of the piston is 4 in., find the tension of the piston-rod when the pump is full of water.

5. The diameter of the piston of a common pump is 6 cm., and the tension of the piston-rod is 13,750 grams when the pump is full of water. Find the distance from the spout to the surface of the water in the well.

6. The tension of the piston-rod of a common pump is 300 pounds when the water in the suction-pipe is 16 feet above the surface of the water in the well. Find the area of the piston.

14. Bramah's Press.

The construction of Bramah's Press is shown in Fig. 112.

It consists of a force pump A, the tube of which opens into a cylindrical vessel B, with very strong, thick sides.

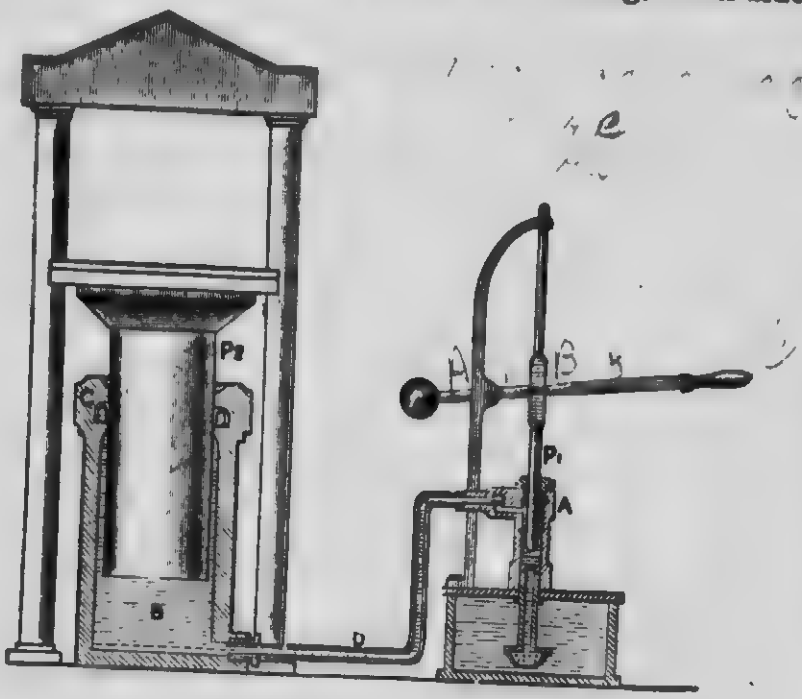

FIG. 112.

In this cylinder there is a large piston or ram P_2, working water-tight in a collar C. A plate to hold the bodies to be pressed is attached to the upper end of the ram. Above this plate is a stationary one supported by the frame-work of the machine. The piston of the pump is worked by a lever.

When the water is pumped into the cylinder the ram P_2 is forced upward and the body is pressed between the

two plates. Since the pressure is transmitted equally in all directions by the water (Art. 5, page 174), the pressure exerted upon the ram P_2 will be as many times that applied to the piston-rod of the pump as the area of a cross-section of the ram P_2 is that of the area of a cross-section of the piston P_1 of the pump. Therefore, by decreasing the size of the piston of the pump, and increasing that of the ram very great pressure may be developed by the machine.

15. Siphon.

The construction of the siphon is shown in Fig. 113.

It consists of a bent tube open at both ends, used for transferring a liquid from one vessel to another. It is filled with the liquid, both branches closed, inverted, and one branch placed in the liquid to be transferred. The end of the other branch must be below the surface of the liquid in the vessel from which the liquid is to be withdrawn.

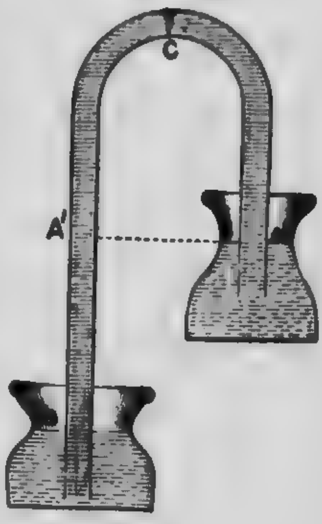

Fig. 113.

When the ends are unstopped, the liquid will run in a continuous stream through the tube.

The pressure at A tending to move the liquid in the siphon in the direction AC

= the atmospheric pressure – the pressure due to the weight of the liquid in AC

and the pressure at B tending to move the liquid in the siphon in the direction BC

= the atmospheric pressure − the pressure due to the weight of the liquid in BC.

But since the atmospheric pressure is the same in both cases, and the pressure due to the weight of the liquid in AC is less than that due to the weight of the liquid in BC, the force tending to move the liquid in the direction AC is greater than the force tending to move it in the direction BC; consequently a flow takes place in the direction ACB. This will continue until the vessel from which the liquid flows is empty, or the liquid comes to the same level in each vessel.

EXERCISE XLI.

1. Is the pressure due to the weight of the water in AC equal to the actual weight of the water in AC? If not, to what is it equal?

2. Upon what does the rapidity of flow in the siphon depend?

3. Will a siphon work in a vacuum? Explain.

4. Upon what does the limit of the height to which a liquid can be raised in a siphon depend?

5. How high can water be raised in a siphon when the mercury barometer stands at 30 in., the sp. gr. of mercury being 13.6?

6. How high can sulphuric acid be raised in a siphon when the mercury barometer stands at 27 in., the sp. gr. of the acid being 1.8 and that of the mercury 13.6?

7. Find the greatest height over which a liquid of density ρ_1 can be carried when the height of the barometer is h, the density of the liquid used in the barometer being ρ.

8. What would be the effect when the siphon is working of making a hole in it (Fig. 113), (1) at C, (2) between A and C, (3) at A^1, (4) between A^1 and C, (5) between A^1 and B?

9. Will any change in the action of a siphon be coincident with a fall in the barometer? Explain.

10. Make a piece of apparatus similar to that shown in Fig. 114, by cutting the bottom off a bottle, bending a glass tube and inserting it into a perforated cork placed in the bottle. Let water from a tap run **slowly** into the bottle. What takes place? Explain.

FIG. 114.

11. Natural reservoirs are sometimes found in the earth, from which the water can run by natural siphons faster than it flows

FIG. 115.

into them from above (Fig. 115). Explain why the discharge through the siphons is intermittent.

Fig 114

ANSWERS.

EXERCISE I. Page 5.

1. $27\frac{3}{11}$; $3\frac{9}{22}$.
2. $48\frac{3}{5}$.
3. 2 : 1 ; 11 : 6.
4. 5 : 56.
5. $\frac{1}{1200}$.
6. $\frac{44}{15}$.
7. 10.
8. (1) 400, (2) 800, (3) 200, (4) 200, (5) 800.
9. 10.
10. 0.98.
11. (1) 3520, (2) $58\frac{2}{3}$, (3) 7040, (4) $29\frac{1}{3}$.
12. 60 miles.
13. 7200.
14. $\frac{15}{22}ab$.
15. $36\frac{ch}{s}$ m.
16. (1) $\frac{5}{44}$, (2) $1\frac{3}{22}$.
17. $\frac{22}{15}\frac{bc}{a}$.

EXERCISE II. Page 9.

1. $4\frac{1}{2}$ ft. per sec.
2. (1) 10.5 cm. per sec., (2) 10 cm. per sec., (3) 11 cm. per sec.
3. (1) 1 cm. per sec., (2) 3 cm. per sec., (3) 1 cm. per sec., (4) 1 cm. per sec.
5. (1) 15 ft. per sec., (2) 75 ft.
6. (1) 35 cm. per sec., (2) 350 cm.
7. (1) 18 cm. per sec., (2) 14 cm. per sec., (3) 56 cm.
8. 175 ft. per sec. ; 150 ft. per sec. ; 125 ft. per sec. ; 100 ft. per sec. ; 540,000 ft.
9. 90 cm. per sec.
10. (1) 435 ft. per sec., (2) 44,250 ft., (3) 126,750 ft.
11. In 10 secs. from the instant its velocity was 20 ft. per sec. ; 1 ft.
12. (1) 4 secs. from the instant it was 8 ft. per sec., (2) 10 secs. from the instant it was 8 ft. per. sec.
13. (1) 50 ft. per sec., (2) 250 ft.
14. 20 secs. ; 800 cm.
15. (1) 5 cm. per sec. per sec., (2) 1750 cm.
16. $55\frac{5}{9}$.

EXERCISE III. Page 13.

1. 600 units of velocity ; 600.
2. 600 ft. per sec ; 600.
3. (1) 300 cm. per sec., (2) 18,000 cm. per sec.
4. (1) $\frac{1}{5}$ ft. per sec., (2) $\frac{1}{300}$.
5. (1) 0.5 ft. per sec., (2) $\frac{1}{120}$.
6. 10 minutes.
7. 1 sec.
8. (1) 6, (2) 2, (3) $\frac{1}{10}$, (4) $\frac{1}{30}$.
9. (1) 50, (2) 5000, (3) $\frac{5}{6}$, (4) $83\frac{1}{3}$.
10. (1) 1, (2) 3600, (3) 3600, (4) 60.

11. (1) 1200, (2) 72,000, (3) 720, (4) 12, (5) $\frac{1}{5}$.
12. (1) 30, (2) 108,000.
13. 38,400.
14. 35,280.
15. 2 : 1.
16. 1000.
17. 1000.
18. 1 : 2.
19. $\frac{1}{2}$.

EXERCISE IV. Page 19.

1. 100 cm. per sec.
2. 20.
3. -185 cm. per sec.
4. (1) 5, (2) 165 cm. per sec., (3) 20 sec. before its velocity was 100 cm. per per sec.
5. (1) 10 sec., (2) $3\frac{1}{3}$ sec.
6. (1) 550 cm., (2) 1 sec.
7. (1) 1.5 sec., (2) 11.25 cm. from starting point.
8. (1) 160 ft., (2) 250 ft., (3) 90 ft.
9. 156 ft.
10. 20 ft. per sec. per sec.
11. (1) 12, (2) 78 ft.
12. (1) 8 ft. per sec. per sec., (2) 10 ft. per sec., (3) 62 feet.
13. (1) 20 cm. per sec., (2) -5 cm. per sec. per sec.
14. 6 ft. per sec. per sec.
15. -32 ft. per sec. per sec.
16. 2 sec ; $\frac{7}{8}$ sec.
17. 16.12 sec. or 81.88 sec.
18. 40 ft. per sec. ; 5 sec.
19. (1) 20 cm. per sec., (2) 4 sec.
20. (1) 20 ft. per sec. per sec., (2) 200 ft. per sec.
21. (1) 6 sec., (2) 144 cm.
22. 44 ft.
23. (1) 40 ft. per sec., (2) 35 ; 25 ; 15 ; 5 ft. per sec.
24. 2 : 1.
25. $23\frac{1}{7}$ ft. per sec.
26. 20 cm. per sec. ; 20 cm. per sec. per sec.
27. Yes, if the body starts from rest.
28. 18 sec. from the time the first particle was at the given point ; 72 m. from the given point.

EXERCISE V. Page 34.

1. 7 : 15.
2. (1) a : 1, (2) 1 : a.
3. a : 1.
4. 33 : 25.
5. Forces are equal.
6. (1) 200 dynes, (2) 25,000 dynes, (3) 30,000 dynes, (4) $55\frac{5}{9}$ dynes, (5) 30 dynes, (6) $\frac{av}{t}$ dynes.
7. (1) 1 cm. per sec. per sec., (2) $\frac{3}{1000}$ cm. per sec. per sec., (3) 1960 cm. per sec. per sec.
8. (1) $\frac{1}{5}$ gram, (2) $2\frac{1}{2}$ grams. (3) 216 grams, (4) 3920 kilograms.
9. (1) 15 cm. per sec. ; $37\frac{1}{2}$ cm., (2) 102 cm. per sec. ; 3468 cm., (3) 4.9 kilometres per sec. ; 24.5 kilometres.
10. 1 hr. 23 min. 20 sec.

EXERCISE VI. Page 41.

1. (1) 160 ft. per sec., (2) 320 ft. per sec.
2. (1) 420 ft. per sec., (2) 260 ft. per sec.
3. (1) 1960 cm. per sec., (2) 980 cm. per sec.
4. (1) 256 ft., (2) 112 ft., (3) $156\frac{1}{4}$ ft.
5. 49 m.
6. $156\frac{1}{4}$ ft.
7. (1) 400 ft., (2) 16 ft.
8. 25 ft.
9. 100 m.
10. (1) $1\frac{1}{2}$ sec. and $4\frac{1}{2}$ sec., (2) 3 sec.
11. (1) $2\frac{1}{2}$ sec., (2) $4\frac{1}{2}$ sec.
12. (1) 6 sec., (2) 5 sec.
13. (1) 96 feet per sec., (2) 126 ft. per sec., (3) 80 ft. per sec.
14. (1) 36 ft. per sec., (2) 20 ft. per sec.
15. (1) $39\frac{1}{16}$ ft., (2) 116.49 ft. per sec.
16. 7 m. per sec.
17. (1) 29.4 m. per sec., (2) 44.1 m. per sec.
18. 64 ft. per sec. ; 5 sec.
19. (1) 50 ft. per sec., (2) 150 ft.
20. 6 sec. ; 176.4 m. from point of projection.
21. $6 \pm 3\sqrt{3}$ sec.
22. 12 sec.
23. (1) 204 ft., (2) 3 sec.
24. 1.547 sec. ; 38.28 ft.
25. 4 sec. more.
26. In [illegible] sec. from instant of projection.
27. (1) 114 ft., (2) 144 ft.
28. $67\frac{1}{3}$ ft.
29. 522 ft.
30. 224 ft.
31. 160 ft.
32. 2 sec.
34. 1936 ft.
35. 1200 ft. per sec.
36. $\frac{b^2}{4a}$ ft.
37. $n \pm \sqrt{n^2 - 2n}$ sec.; $g\sqrt{n^2 - 2n}$.

EXERCISE VII. Page 45.

1. (1) 9,800,000 dynes, (2) $\frac{1}{98}$ ~~dynes~~ grms.
2. 3750 : 49.
3. $\frac{am}{980}$ grams.
4. 10 min.
5. 37.5 dynes.
6. 785 cm. per sec. per sec.
7. 7,350,000 units.
8. Forces are equal.
9. 10 cm. per sec. per sec.
10. 2 : 1 ; 1 : 2.
11. 512 cm.
12. 2 sec., 32 cm. per sec.; 4 sec., -32 cm. per sec.
13. 900 dynes.
14. 5 : 98 ; 5 metres per sec.
15. 54 dynes.
16. 12 dynes.
17. 9800 units ; $181\frac{13}{27}$ cm. ; $21,777\frac{7}{9}$ cm.
18. 36 cm. per sec. ; 320 dynes.
19. 300 dynes.
20. 49 kilograms.
21. 20 grams.
22. 5 : 7.
23. 1 : 490.
24. (1) 19,600 dynes, (2) 9,613,800 dynes.
25. 98 cm. per sec. per sec.
26. 490.5 grams.
27. 107,800 dynes.
28. (1) 117,600,000 dynes, (2) 39,200,000.

EXERCISE VIII. Page 50.

1. 80 cm. per sec. per sec.; 144,000 dynes.
2. 2520 cm.
3. $\frac{4}{7}$ sec.; 56 cm. per sec.
4. (1) 1750 cm., (2) 1050 cm., (3) 700 cm. per sec.
5. 709.1 cm. per sec.
6. 8960 cm.; 1120 cm.
7. 14.01 cm. per sec.
8. 7056 dynes.
9. (1) 7750 cm., (2) 2790 cm.
10. 985 grams.
11. 5 : 3.
12. 5 : 3.
13. $\frac{1}{10}$ m.
14. $7\frac{5}{11}$ sec.; $664\frac{16}{121}$ cm.
16. $\dfrac{4\,ag}{(g+a)^2}m$.
17. 96,000 dynes.
18. 1960 dynes.
19. 950 cm. per sec per sec.
20. 490 cm.
21. $\dfrac{g}{\sqrt{3}}$.

EXERCISE IX. Page 55.

1. 940.8×10^7 ergs.
2. 1,000 ergs.
3. 98,000 joules.
4. 98,000 joules.
5. 3,920 joules.
6. 144 joules.
7. 21.315 joules.
8. 1,886,500 joules.
9. 10 metres.
10. 1509.2 joules.

EXERCISE X. Page 57.

1. (1) 4,802,000 ergs, (2) 1,200,500 ergs, (3) 0, (4) 4,802,000 ergs.
2. 12,500 joules.
3. 7,203 joules.
4. 8,000 joules.
5. 2,500 joules.
6. 20 joules.
7. 9.8 joules.
9. (1) 9.8 joules.
10. 832×10^5; $11,648 \times 10^7$.
11. (1) Forces are equal, (2) in ratio $m : M$.
12. Velocity in the ratio 4.3; mass in the ratio 9.4.
13. 346.4 cm. per sec.
14. 18 cm.
15. 458.25 metres per sec.
17. 15 units of weight; 8 units of velocity.
18. (1) 16,611,000 units of momentum, (2) 8,139.39 joules.

EXERCISE XI. Page 60.

1. 60 erg-seconds.
2. 100 erg-seconds.
3. 10,000 erg-seconds.
4. 100.
5. 1,000.
6. 100.
7. 20.
8. 39.2.
9. 0.28 horse-power.
10. 784 horse-power.
11. 980.
12. 70 watts.
13. 196.
14. $192\frac{36}{175}$.
15. 6,000,000.
16. 600 litres.

EXERCISE XIII. Page 69.

1. 35 gm.; 5 gm.
2. 2 P; 2 Q.
3. 39 pds.
4. 37 kgm.
5. 18 pds.
6. 12 P.
7. 12 pds. and 16 pds.
8. 20 gm. and 48 gm.
9. 6 pds. and $3\sqrt{5}$ pds.
10. 15 pds. and 20 pds.
11. 10 gm. and 24 gm.
12. 13 pds.
13. 15 pds.
14. 400 pds.

EXERCISE XIV. Page 71.

1. (1) 84 pds., (2) 18.477 pds., (3) 5.176 pds., (4) 70 pds., (5) 8.789 pds., (6) 2.125 pds., (7) 18.915 pds., (8) 12.64 pds., (9) P pds. north.
3. 17 pds.
4. 20 pds.
5. Forces are equal.
6. 8 gm.
7. 12 pds.
8. 12 pds. and 20 pds.
9. 10 pds. and $10\sqrt{2}$ pds.
10. 3 gm.; 1 gm.
11. 2 : 1.
12. $\sqrt{6}$ pds.
13. 50 pds. acting toward the centre.
14. $5\sqrt{2}$ kgm. at 135° with first force.
15. 5 : 4.
16. 10 and 26 pds.
17. $\frac{1}{2}\sqrt{7}$ times the side of the triangle.
26. Resultant is represented by $OD = 2\sqrt{3}$ pds.
31. (1) $\sqrt{3}$ pds., (2) Resultant is represented by AD.

EXERCISE XV. Page 78.

1. (1) $5\sqrt{3}$ pds., (2) $5\sqrt{2}$ pds., (3) 2.58 pds.
2. $10\sqrt{3}$ and 10 pds.
3. $6\sqrt{2}$ pds.
4. $50\sqrt{2}$ pds.
5. $8\sqrt{3}$ and 8 pds.
6. $\frac{4}{3}\sqrt{3}$ pds.
7. 199.23 pds.
8. 98.48 pds.
11. (1) 6 pds., (2) 6 pds.

EXERCISE XVI. Page 82.

1. $3\sqrt{3}$ pds.
2. 22.2 or 10.56 pds.; makes with the first force an angle whose tangent is $\frac{3}{8}\sqrt{3}$ or $\frac{1}{2}\sqrt{3}$.
3. 14 pds.
4. 28 pds. nearly.
5. $3\sqrt{3}$ pds. at 30° with third force.
7. 7.464 pds.
8. 5.477 pds.
9. 10 P toward opposite angular point.
10. Equals one of the forces in the direction of the sixth side.

EXERCISE XVII. Page 96.

1. 40 and $20\sqrt{3}$ pds.
3. $1:1:\sqrt{3}$.
4. 5 and $5\sqrt{3}$ pds.
5. 10 and 20 pds.
6. $2:1$.
7. $\dfrac{10\sqrt{2}}{1+\sqrt{3}}$ and $\dfrac{20}{1+\sqrt{3}}$ pds.
8. $\dfrac{\sqrt{2}}{3}$ pds. $\dfrac{2}{\sqrt{3}}$
9. $4\sqrt{3}$ pds.
10. $3\sqrt{2}$ pds.
11. 100 pds.
12. $10\sqrt{3}$ and 10 pds.
13. (1) $20\sqrt{3}$ pds., (2) 40 pds.
14. $\dfrac{100}{\sqrt{3}}$ pds. ; $\dfrac{100}{\sqrt{3}}$ or $\dfrac{200}{\sqrt{3}}$ pds.
15. (1) $\sqrt{3}$ pds., (2) 1 and $\sqrt{3}$ pds., (3) 45°.
16. 392 cm. per sec. per sec.; 7056 dynes.
17. 350 cm.
18. $233\frac{1}{3}$ cm. per sec. per sec.
19. $140\sqrt{2}$ cm. per sec.; $\dfrac{5\sqrt{2}}{28}$ seconds.
20. 30°.
21. $\sin^{-1}(\frac{1}{20})$.
22. $\frac{20}{29}$ metres.
23. (1) 1,200,500 ergs, (2) 9,800 units of momentum, (3) in 4 seconds.

EXERCISE XVIII. Page 97.

4. 120° between any two forces.
6. (1) 120°, (2) 90°.
7. $1:1:\sqrt{3}$.
8. 6 gm., $6\sqrt{3}$ gm.
9. $5\frac{1}{3}$ pds. ; $6\frac{2}{3}$ pds.
10. $7\frac{1}{2}$ pds. ; $12\frac{1}{2}$ pds.
11. $\dfrac{20}{\sqrt{3}}$ pds.
12. 38.4 pds. ; 28.8 pds.
13. 60 pds. ; 25 pds.
14. 74 pds.
15. $6\frac{2}{3}$ pds.
16. 6 pds.
17. 6 pds. ; 8 pds.
18. 5 pds. ; 13 pds.
19. 12 pds.
20. $\frac{1}{3}\sqrt{3}$ pds. A
28. X lies in EA produced so that AX = AE.

EXERCISE XIX. Page 107.

1. 5 dynes acting 3 metres from smaller force.
2. 8 dynes acting 25 metres from smaller force.
3. (1) 8 pounds, $7\frac{1}{2}$ ft. from 7 pound force, (2) 22 pounds $2\frac{4}{11}$ ft. from 7 pound force.
4. $70\frac{7}{12}$ pounds, $50\frac{5}{12}$ pounds.
5. $37\frac{1}{3}$ pounds, $74\frac{2}{3}$ pounds.
6. 24 and 16 pounds.
7. 2 ft. from the stronger man.
8. $7\frac{1}{2}$ ft. from fixed point.
9. 40 inches from greater force.
10. 8 dynes and 6 dynes.
11. $6\frac{1}{4}$ pounds ; $3\frac{3}{4}$ pounds.
12. $\dfrac{mP}{m+n}$, $\dfrac{nP}{m+n}$; $\dfrac{ma}{n}$.
13. 35 and 40 pounds.
14. 42 and 21 pounds.
15. $11\frac{3}{7}$ inches and $7\frac{5}{7}$ inches.
16. 17.5 cm.

EXERCISE XX. Page 110.

1. (1) 0, -6; (2) 18, 18; (3) 0, 0; (4) 0, $-11\sqrt{2}$; (5) $\sqrt{2}$, 0; (6) 40, 0.
2. 0; 108; -108.
3. $30\sqrt{3}$.
4. (1) 201.47, (2) $62\frac{1}{2}$.
5. 0; 160; 0; -160.
6. $25\sqrt{2}$ ft. from the ground.
7. (1) 66.352, (2) 79.672.
8. $\frac{7}{13}$ metres from A.
9. 1 : 2.
10. 10 cm. from B.

EXERCISE XXI. Page 118.

1. 6.6 metres from the 20 kilogram mass.
2. $1\frac{1}{3}$ ft. from fulcrum.
3. $4\frac{2}{7}$ ft. from same end.
4. $1\frac{3}{8}$ ft. from 7 lb. mass.
5. 2 grams.
6. $267\frac{2}{3}$ pounds; $624\frac{1}{3}$ pounds.
7. 27 dynes at a point $1\frac{23}{27}$ metres from end.
8. 6 dynes.
9. $11\frac{1}{7}$ cm. from 3 dyne force.
10. 5 lbs.
11. 35 lbs.; 40 lbs.
12. $11\frac{3}{7}$ inches; $7\frac{5}{7}$ inches.
13. B is 3 cm. from nearest peg.
14. One quarter of the length of the beam.
15. 11 ft. from smaller end.
16. $1\frac{1}{2}$ inches.
17. $2\frac{4}{15}$ metres from end; 1 kgm.; 2 kgm.; 4 kgm.; 8 kgm.
18. $\frac{W}{2}\left(\frac{W_1+W_2-2W_0}{W_0-W_1}\right)$.
19. $3\frac{1}{3}$ grams; $8\frac{1}{3}$ cm. from 5 gram mass.
20. 12 cwt.

EXERCISE XXII. Page 126.

1. 30 lbs.; 60° with rod.
2. $\frac{1}{2}$ kgm.; $\frac{\sqrt{3}}{2}$ kgm.
3. $10\sqrt{3}$ lbs.
4. 120 lbs.
5. $\frac{\sqrt{13}.W}{2\sqrt{3}}$; $\frac{W}{2\sqrt{3}}$.
6. 30°.
8. 12 pounds; $6\sqrt{3}$ pounds.
9. 1 : 2.
10. 30 pounds; 36.61 pounds.
11. (1) $\frac{100}{\sqrt{3}}$ pounds; (2) $\frac{100}{\sqrt{3}}$ pounds: (3) 200 pounds.
12. $10\sqrt{3}$ pounds.
13. $30\sqrt{3}$ pounds; $30\sqrt{39}$ pds.
14. $\frac{2}{3}W\sqrt{3}$; $\frac{1}{3}W\sqrt{3}$.
15. (1) $22\frac{1}{2}$ pounds; (2) 54.8 pounds nearly.
16. $46\frac{2}{3}$ pounds; 68.4 pounds.
17. W; P.
18. 3 ft. above A; $\frac{200}{\sqrt{3}}$ pounds; 30° with the wall.
19. 3 : 1
20. $25\sqrt{3}$; $25\sqrt{7}$.
21. 15 kgm.; 10 kgm.; $5\sqrt{3}$ kgm.
22. 25 pounds; $25\sqrt{2}$ pounds.
23. $\tan^{-1}\frac{11}{7}\sqrt{3}$; $\frac{11}{10}\sqrt{309}$ pds.
25. 30 pounds; $30\sqrt{3}$ pounds.
26. 30 kgm.; $30\sqrt{2}$ kgm.
27. a; $\frac{2W}{\sqrt{3}}$.
28. $30\sqrt{3}$ pounds.
29. 10 kgm.

30. (1) $\frac{W}{\sqrt{3}}$, (2) $\frac{2W}{\sqrt{3}}$.

31. (1) 30°, (2) $6\sqrt{3}$ kgm., (3) $3\sqrt{3}$ kgm.

32. $3\sqrt{3}$ pounds.

33. $6\sqrt{3}$ pounds; $3(3\sqrt{2}-\sqrt{6})$ pounds.

34. $W(\sqrt{2}-1)$.

35. 11 : 3.

37. 20 pounds; $20\sqrt{3}$ pounds; 30°.

38. 27 pounds; 28.9 pounds.

39. (1) $4\frac{8}{37}$ pounds, (2) 10 pounds, (3) $4\frac{8}{37}$ pounds.

40. 120 pounds; 0.

41. $50\sqrt{7}$ pounds.

42. $\sqrt{3}$ tons.

EXERCISE XXIII. Page 136.

1. $2\frac{2}{3}$ cm.; 3.283 cm.

3. 2 ft.; $3\frac{1}{3}$ ft.; $3\frac{1}{3}$ ft.

4. $\frac{2}{3}$ ft.; 1 ft.; $\frac{2}{3}$ ft.

5. $3\frac{1}{3}$ ft.; 5.696 ft.; 4.807 ft.

6. 10 inches.

EXERCISE XXIV. Page 137.

1. 18.04 inches.

2. At the centre of the base of the triangle.

3. $7\frac{3}{5}$ inches.

4. $\frac{11}{30}$ of the side of the square from the middle point of the base.

5. $\sqrt{3}:1$.

6. $1\frac{1}{2}$ inches.

8. $2\frac{1}{6}\,a$ from the foot of the cross (a = side of the square).

9. $2\frac{1}{14}$ inches from the joint.

10. $5\frac{7}{8}$ inches from the middle point of the lower side of the figure.

11. $\frac{12a^2+6ab+b^2}{3(4a+b)}$.

12. Divides diagonal of larger square in the ratio 7 : 13.

13. Divides perpendicular from right angle on hypotenuse in the ratio 26 : 1.

15. In the line joining the middle points of the 6-in. and 14-in. sides, at a distance of $\frac{26\sqrt{3}}{15}$ in. from the latter.

16. $3\frac{1}{8}$; $2\frac{1}{3}$.

20. $11\frac{1}{3}$ inches.

23. $\frac{23}{48}$ of BD from D.

EXERCISE XXV. Page 140.

1. $\frac{a}{9}$ from E.

2. $\frac{3}{8}$ height from base.

3. $OG=\frac{2}{21}\,OC$.

4. In the straight line drawn parallel to BC from the middle point of AB and at a distance $\frac{26}{45}$ of the side of the square from this point.

5. Distances from AD and AB are $\frac{1}{18}$ AB and $\frac{4}{9}$ AD.

6. In the diameter of the rectangle parallel to side a and at a distance $\frac{1}{6}\,a$ from G.

7. $\frac{7\sqrt{3}}{5}$ inches.

8. $\frac{5}{12}\,a$.

11. $\frac{2}{3}$ of the line from middle of the base to the vertex.
12. $\frac{4}{15}$ of the median from the base.
13. $\frac{11}{12}$ of the diagonal from that corner.
14. $\frac{7}{15}$ of the median from the base.
15. In line joining their centres at a distance of 1 ft. $8\frac{1}{4}$ inches from the centre of the hole.
16. In line joining centres 2 inches from the centre of the larger circle.
18. Centre of hole 16 inches from centre of disc.
19. Distant $\frac{\sqrt{2}}{14}$ of the radius bisecting the angle between the two radii from the centre.
20. $\frac{14}{171}$ inches from centre of plate in line joining centre of plate with centre of hole.

EXERCISE XXVI. Page 143.

1. 6 inches.
2. 10 inches from the 12 lb. mass.
3. $4\frac{2}{3}$ inches from the end.
4. $8\frac{1}{3}$ inches from the 7 lb. mass.
5. 15 inches from end.
6. $28\frac{4}{5}$ ft. from first man.
7. $6\frac{3}{4}$ feet from 12 lb. mass.
8. $3\frac{1}{4}$ feet from 1 lb. mass.
9. 3.26 in. from the top.
10. 3.3 inches from the base.

EXERCISE XXVII. Page 146.

1. $\frac{2}{3}$ of diagonal from 2 lb. mass.
2. OG $= \frac{1}{4}$ OD.
3. 4.34 inches.
4. $\frac{3}{4}$ of the side of the square.
5. 3.6 feet nearly.
6. 7.8 inches nearly.
7. $8\frac{1}{3}$ in. ; $11\frac{1}{3}$ in.
16. On the diameter of the circle drawn from angular point at which no weight is placed at a distance $\frac{3}{14}$ of diameter from that point.
17. 9 inches.

EXERCISE XXVIII. Page 152.

5. 60°.
7. 3.
9. $3\frac{11}{13}$ ft.
10. 120.
11. 10.
13. $3\sqrt{3}$ feet.
15. $5(\sqrt{3}-1)$ cm.
16. $\text{Tan}^{-1}\frac{1}{4}$.
17. $\text{Tan}^{-1}\frac{3}{4}$; $\tan^{-1}\frac{4}{3}$.
18. 10 kgm.
19. 50 pounds.
20. 120 pounds.
21. $\text{Tan}^{-1}\frac{3}{13}$.
24. $\frac{W}{6}$.
25. $a\sqrt{3}$ where $a =$ side of square.

EXERCISE XXIX. Page 156.

1. 6 pds.
2. $10\sqrt{3}$ pds.
3. 5 gm.
4. 13 pds
5. (1) 30 pds., (2) $30\sqrt{2}$ pds., (3) $30\sqrt{3}$ pds.
6. (1) 8 pds., (2) 32 pds.
7. 60 pds.
8. (1) $(50\sqrt{3}-5)$ pds., (2) $(50\sqrt{3}+5)$ pds.
9. (1) $2\frac{1}{2}$ pds., (2) $7\frac{1}{2}$ pds., (3) $(\frac{5}{2}\sqrt{15}-2\frac{1}{2})$ pds.

EXERCISE XXX. Page 167.

1. 2 pds. ; 10.196 pds.
2. $\frac{1}{15}$.
3. 4.714.
4. $\sqrt{3}$.
5. $\sqrt{3}$; 1 ; $\frac{\sqrt{3}}{3}$.
6. $\frac{1}{\sqrt{3}}$.
7. 11.732 pds.
8. 36 pds.
9. 10 pds.
10. .732.
11. $24\frac{1}{4}$ pds.
12. $\frac{1}{2}\sqrt{3}$ pds.
13. .268.
14. 30°.
17. $g(\sin\theta - \mu\cos\theta)$.

EXERCISE XXXI. Page 176.

1. (1) 4, (2) 576.
2. (1) 6, (2) 54.
3. (1) 50, (2) 500,000.
4. (1) 0.2, (2) 0.8.
5. 80 pds.
6. $\frac{2}{11}a$.
7. $30\frac{1}{4}$ pds.
8. 20 kgm.
9. $31\frac{1}{17}$ gm.
10. 1.5 sq. m.
11. $\frac{31}{50}$ in.
12. 35437.5 gm.
13. $1919\frac{3}{8}$ pds.
14. 7 kgm.

EXERCISE XXXII. Page 186.

1. 9.122 pds.
2. 0.0375 gm.
3. 11.5 pds.
4. 2.3 gm. per sq. mm.
5. 8 : 9.
6. 1 : 100.
7. 5 : 2 : 3.
8. 4 : 5.
9. 60 ft.
10. 9 m.
11. $31\frac{1}{4}$.
12. 9 kgm.
13. (1) $230\frac{3}{8}$ ft., (2) $1\frac{3}{8}$ ft.
14. 30 ft.
15. 1360.
16. 73.53 cm.
17. 4100.
18. 125 pds.

EXERCISE XXXIII. Page 191.

1. 5000 pds.
2. 30,000 kgm.
3. 3750 pds.
4. 107,500 kgm.
5. 77,000 pds.
6. 37,500 pds.
7. 72,000 kgm.

8. (1) 250,000 pds., (2) 62,500 pds., (3) 187,500 pds.
9. (1) 147,000 pds., (2) 48,562$\frac{1}{2}$ pds., (3) 84,000 pds., (4) 63,000 pds.
10. (1) 42,000 pds., (2) 3600 pds., (3) 24,000 pds.
11. (1) 24,000 kgm., (2) 6000 kgm., (3) 10,500 kgm.
12. 182,000 kgm.
13. On each vertical face 187.5 pds.; on upper face 156.25 pds.; on lower face 218.75 pds.
14. (1) 212,500 pds., (2) 137,500 pds., (3) 87,500 pds.
15. 27 kgm.
16. 1500 pds.
17. 31.25 pds.
18. 100 ft.
19. 8 m.
20. 288 ft.
21. Upper side $2\frac{1}{8}$ in. below surface.
22. 7.25 ft.
23. 15 in.; top, 4.88 pds.; vertical side, 5.37 pds.; bottom, 5.859 pds.
24. 2 : 1.
25. (1) 2.4 gm., (2) 0.64 gm., (3) 0.48 gm.
26. 2 : 1.
27. 4 : 13.
28. (1) 3080 kgm., (2) 8800 kgm.
29. (1) 11,160 gm., (2) 92,342$\frac{6}{7}$ gm.
30. (1) 9000 pds., (2) 2500 pds., (3) 3000 pds.
31. (1) 905$\frac{1}{7}$ gm., (2) 1005$\frac{5}{7}$ gm., (3) 301$\frac{5}{7}$ gm.
32. 193·526 gm.
33. (1) 1848 kgm., (2) 792 kgm., (3) 1760 kgm.
34. 817$\frac{1}{7}$ kgm.
35. 8.
36. 18 in.
37. 400 gm.
38. (1) 416 kgm., (2) 1104 kgm., (3) 276 kgm., (4) 319.8 kgm., (5) 96.2 kgm.
39. 16 : 39.
40. (1) 14 in., (2) 64 in.
41. $2\sqrt{2}$ in. from the surface of the liquid.

EXERCISE XXXIV Page 197.

1. 10 in.
2. 68 cm.; 170 cm.; 255 cm.
3. 2.427 ft.
4. 13.619.
5. 11 : 7.
6. 15 cm.
7. 14.96 cu. in.
8. $\frac{5}{187}$ of height of one arm.
9. Water 6.047 in.; alcohol 5.953 in.
10. $4\frac{1}{2}$ in.
11. At the bottom of the vertical tube containing the oil.

EXERCISE XXXV. Page 201.

1. (1) 62.5 pds., (2) 237.5 pds.
2. 4.96 pds.
3. 2.5 kgm.
4. 5 gm.
5. 295 pds.
6. 7 kgm.
7. 6.5 kgm.; 19,993.5 kgm.
8. 968.75 pds.
9. 10 kgm.
10. $n + 108\,pm$.
11. 14,000 kgm.
12. 100 pds.

13. 2 c.dcm.
14. 6 cu. ft.
15. 20 cm.
16. 16 oz. ; 12 cu. in.
17. (1) 100.078 gm., (2) 60 c. cm.
18. 159.14 gm.
19. 311.9 gm.
20. 28.5 gm.
22. 1.5 gm. placed with the weights.
24. (1) $1\frac{1}{3}$ c.cm., (2) 1 c.cm.

EXERCISE XXXVI. Page 204.

1. 0.413 oz.
2. $\dfrac{n-m}{n}$ gm.
3. 0.881.
4. 5 c.dcm.
5. $4\frac{1}{8}$ cu. ft.
6. $\frac{1}{3}$.
7. 5.64.
8. $\frac{1}{3}$.
9. 225 pds.
10. n^2 c.cm.
11. $32\frac{1}{3}$ pds.
12. 520 gm.
13. 42 gm.
14. 9 pds.
15. 430 pds.
16. 41.56 in.
17. 216 sq. in.
18. 20 lbs.
19. 125,000.
20. Forces are equal.
21. $40\frac{5}{8}$ lbs.
22. $13\frac{1}{3}$ lbs.
23. (1) 30 gm., (2) 20 gm.
24. 1000.
25. 3 cm. edge.
26. 1 : 6.
27. 10.65 gm. per c.cm.
28. 0.536 gm. per c.cm.
29. 4.75 in.
30. 23.48 cm.
31. 10,000 cu. ft.
32. 0.514.
33. 15 oz.
34. 0.64 in.
35. 30 lbs.
37. 1.458 pds.
38. (1) 2.604 pds., (2) 2.007 pds.
39. 7.102 pds. ; 50 pds.

EXERCISE XXXVII. Page 216.

1. 14.756 pds.
2. 1033.6 gm.
3. (1) 15 pds., (2) $14\frac{1}{3}$ pds., (3) 438.48 gm.
4. (1) 952 gm., (2) 1034 gm., (3) 1030 gm.
5. $7083\frac{1}{3}$ pds.
6. 24 ft. ; $9\frac{3}{8}$ ft. ; $7\frac{1}{3}$ ft.
7. 76 cm. ; 152 cm. ; 304 cm.
8. 272 in. ; 240 in. ; $81\frac{3}{8}$ in.
9. 1033.6 cm.
10. 1.43.
11. 1307.9 oz. per cu. ft.
12. 118.23 in.
13. $56\frac{3}{8}$ ft.
14. 12.5 : 1.
15. (*a*) 1, 22,000 pds. ; 8, (1) 1,100,000 pds., (2) 487,500 pds., (3) 612,500 pds. ; 13, $2{,}312\frac{1}{2}$ pds. on vertical face, $2{,}281\frac{1}{4}$ pds. on upper face, $2{,}343\frac{3}{4}$ pds. on lower face ; 18, 66 ft.
(*b*) 2, 92,016 kgm., 28, (1) 18,997.44 kgm. ; (2) 99,756.8 kgm.
16. $2833\frac{1}{3}$.
17. 19.87 ft.

EXERCISE XXXVIII. Page 221.

1. (*a*) 70 cm.
   (*b*) 35 cm.
2. 6 cm., 4 cm., 3 cm.
4. $6\frac{2}{3}$ gram per sq. cm.
5. 175 c.cm.
6. $63\frac{1}{3}$ cm.
7. 21.22 c.cm.
8. 50 cm. ; 50 gram.

EXERCISE XXXIX. Page 224.

1. $(5)^5 : (6)^5$.
2. (1) $(\frac{4}{5})^3$, (2) $(\frac{4}{5})^5$, (3) $(\frac{4}{5})^6$.
3. 3.
4. (1) $(\frac{3}{4})^5 \times 30$ in., (2) $(\frac{3}{4})^{15} \times 30$ in., (3) $(\frac{3}{4})^{20} \times 30$ in.
5. 4.
6. 11 : 1.
7. 3 : 1.
8. 75.12 gm.
10. 3 : 2.
11. 10.

EXERCISE XL. Page 229.

1. 10.336 m.
2. 17 ft.
3. $20\frac{5}{8}$ pds.
4. $114\frac{7}{12}$ pds.
5. $486\frac{1}{9}$ cm.
6. $43\frac{1}{5}$ sq. in.

EXERCISE XLI. Page 232.

5. 34 ft.
6. 17 ft.
7. $h\dfrac{\rho}{\rho_1}$.

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